Operations Research I

STA 343

University of Ibadan Distance Learning Centre

Open and Distance Learning Course Series Development

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Vice-Chancellor's Message

The Distance Learning Centre is building on a solid tradition of over two decades of service in the provision of External Studies Programme and now Distance Learning Education in Nigeria and beyond. The Distance Learning mode to which we are committed is providing access to many deserving Nigerians in having access to higher education especially those who by the nature of their engagement do not have the luxury of full time education. Recently, it is contributing in no small measure to providing places for teeming Nigerian youths who for one reason or the other could not get admission into the conventional universities.

These course materials have been written by writers specially trained in ODL course delivery. The writers have made great efforts to provide up to date information, knowledge and skills in the different disciplines and ensure that the materials are user-friendly.

In addition to provision of course materials in print and e-format, a lot of Information Technology input has also gone into the deployment of course materials. Most of them can be downloaded from the DLC website and are available in audio format which you can also download into your mobile phones, IPod, MP3 among other devices to allow you listen to the audio study sessions. Some of the study session materials have been scripted and are being broadcast on the university's Diamond Radio FM 101.1, while others have been delivered and captured in audio-visual format in a classroom environment for use by our students. Detailed information on availability and access is available on the website. We will continue in our efforts to provide and review course materials for our courses.

However, for you to take advantage of these formats, you will need to improve on your I.T. skills and develop requisite distance learning Culture. It is well known that, for efficient and effective provision of Distance learning education, availability of appropriate and relevant course materials is a *sine qua non.* So also, is the availability of multiple plat form for the convenience of our students. It is in fulfilment of this, that series of course materials are being written to enable our students study at their own pace and convenience.

It is our hope that you will put these course materials to the best use.

Prof. Abel Idowu Olayinka

Vice-Chancellor

Foreword

As part of its vision of providing education for "Liberty and Development" for Nigerians and the International Community, the University of Ibadan, Distance Learning Centre has recently embarked on a vigorous repositioning agenda which aimed at embracing a holistic and all encompassing approach to the delivery of its Open Distance Learning (ODL) programmes. Thus we are committed to global best practices in distance learning provision. Apart from providing an efficient administrative and academic support for our students, we are committed to providing educational resource materials for the use of our students. We are convinced that, without an up-to-date, learner-friendly and distance learning compliant course materials, there cannot be any basis to lay claim to being a provider of distance learning education. Indeed, availability of appropriate course materials in multiple formats is the hub of any distance learning provision worldwide.

In view of the above, we are vigorously pursuing as a matter of priority, the provision of credible, learner-friendly and interactive course materials for all our courses. We commissioned the authoring of, and review of course materials to teams of experts and their outputs were subjected to rigorous peer review to ensure standard. The approach not only emphasizes cognitive knowledge, but also skills and humane values which are at the core of education, even in an ICT age.

The development of the materials which is on-going also had input from experienced editors and illustrators who have ensured that they are accurate, current and learner-friendly. They are specially written with distance learners in mind. This is very important because, distance learning involves non-residential students who can often feel isolated from the community of learners.

It is important to note that, for a distance learner to excel there is the need to source and read relevant materials apart from this course material. Therefore, adequate supplementary reading materials as well as other information sources are suggested in the course materials.

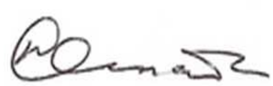
Apart from the responsibility for you to read this course material with others, you are also advised to seek assistance from your course facilitators especially academic advisors during your study even before the interactive session which is by design for revision. Your academic advisors will assist you using convenient technology including Google Hang Out, You Tube, Talk Fusion, etc. but you have to take advantage of these. It is also going to be of immense advantage if you complete assignments as at when due so as to have necessary feedbacks as a guide.

The implication of the above is that, a distance learner has a responsibility to develop requisite distance learning culture which includes diligent and disciplined self-study, seeking available administrative and academic support and acquisition of basic information technology skills. This is why you are encouraged to develop your computer skills by availing yourself the opportunity of training that the Centre's provide and put these into use.

In conclusion, it is envisaged that the course materials would also be useful for the regular students of tertiary institutions in Nigeria who are faced with a dearth of high quality textbooks. We are therefore, delighted to present these titles to both our distance learning students and the university's regular students. We are confident that the materials will be an invaluable resource to all.

We would like to thank all our authors, reviewers and production staff for the high quality of work.

Best wishes.



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General Introduction

Operations research which is concerned with the efficient allocation of scarce resources is both an art and science. The art lies, in the ability to depict the concepts 'efficient and scarce' in all defined mathematical model of a given situation. The science consists in the derivation of computational methods for solving such models.

Since the optimal allocation of money, manpor, energy or a host of other scarce factors, is of importance to decision makers in many traditional disciplines, this book will be useful to individual from a variety of backgrounds.

Therefore this outline has been designed both as an introductory textbook for beginners in operations research and as a reference manual from which practitioners can obtain specific procedures.

This book is divided into the Study Sessions. The first Study Session discusses useful concepts, nature and scope of operations research. Study Session two to eleven focuses on theories and problems while the last Study Session introduces sequencing and scheduling in operations research.

At the end of these Study Sessions, you should be able to:

1. Explain and solve problems relating to some concepts in operations research.
2. Emulate and solve Linear Programming problems and apply them to solve management problems.
3. Discuss the concepts and principles of Sensitivity Analysis.
4. Discuss the concepts and principles of Duality theory.
5. Formulate and solve Transportation problems
6. Formulate and solve Assignment problems
7. Sketch and solve problems of Network Analysis.
8. Discuss the applications and theories in Inventory
9. Discuss the concepts and principles of Sequencing and Scheduling in operations research.

Study Session One: Nature and Scope of Operations Research

Expected duration: 1 week or 2 contact hours

Introduction

This Study Session will introduce you to the basic terminologies in operations research, including mathematical modeling, feasible solutions, optimization and iterative computations. You will learn that defining the problem correctly is the most important (and most difficult) phase of practicing Operations Research (OR). The Study Session also emphasizes that while mathematical modeling is a cornerstone of OR, unquantifiable factors (such as human behaviour) must be accounted for in the final decision.

Learning Outcome for Study Session 1

At the end of this Study Session, you should be able to:

1.1 Define operations research.
1.2 List the three principal components of a linear programming model.
1.3 Define some basic concepts in OR.
1.4 Discuss the principal phases in practice for implementing OR.
1.5 Discuss briefly some techniques used in OR.

1.1 Definitions of Term

Operations research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defense. It is concerned with an efficient allocation of limited resources to known activities with the objective of meeting a desire goal such as maximization (profit) or minimization (cost). It is a dominant and indispensable decision-making tool.

A common misconception held by many is that O.R. is a collection of mathematical tools. While it is true that it uses a variety of mathematical techniques, operations research has a much broader scope. It is in fact a systematic approach to solving problems, which uses one or more analytical tools in the process of analysis.

Perhaps the single biggest problem with O.R. is its name; to a layperson, the term "operations research" does not conjure up any sort of meaningful image! This is an unfortunate consequence of the fact that the name that A. P. Ro is credited with first assigning to the field was somehow never altered to something that is more indicative of the things that O.R. actually does.

In-Text Question

Operations Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defense. True or false

In-Text Answer

True

Sometimes O.R. is referred to as Management Science (M.S.) in order to better reflect its role as a scientific approach to solving management problems, but it appears that this terminology is more popular with business professionals and people still quibble about the differences between O.R. and M.S. Compounding this issue is the fact that there is no clear consensus on a formal definition for O.R.

For instance, C. W. Churchman who is considered one of the pioneers of O.R. defined it as *the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to problems*.

Algorithm: It is called an iteration which provides fixed computational rules that are applied repetitively to the problem with each repetition moving the solution closer to the optimum.

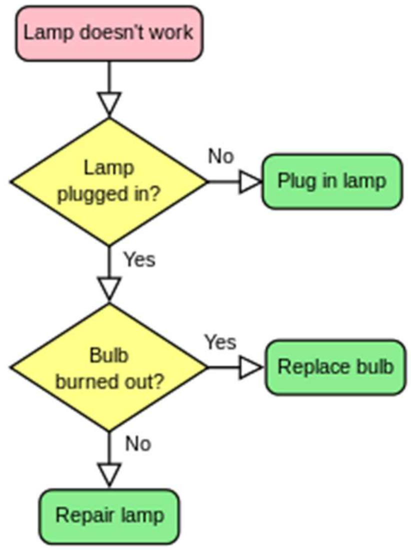

Figure 1.1 An algorithm showing how electric lamp work

Basic variable: It is a variable which is a unit coefficient in an equation (constraint) but set to zero in other equations (constraints). Those variables which are not basic (which are always set to zero) are called Non-basic variables.

Slack variables are non-negative variables use in augmenting or converting an inequality sign (less or equal to) to and equation which are added to the left-hand side of the constraint.

Surplus variables are non-negative variables use to convert an inequality sign (greater or equal to) an equation which are subtracted from the left-hand side of the equation.

Feasible Solution: This is a solution which satisfies all the constraints (conditions in the model). A solution which does not satisfy all the constraints or Linear Programming (LP) model with inconsistent constraints is called an infeasible solution.

Optimal Solution: This is the feasible solution which yields the best (maximization or minimization) value of the objective function.

Unbounded Solution: In some LP models, the value of the variables may be increased indefinitely without violating any of the constraints, meaning that solution space is unbounded in at least one direction. As a result, the objective value may increase (maximization case) or decrease (minimization case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded.

Degeneracy: In the application of the feasibility condition of simplex method (condition for the learning variable), tie for the minimum ratio may be broken arbitrarily. When a tie happens at least one basic variable will be zero in the next iteration and the new solution is said to be degenerate.

Alternative Optima: When the objective function is parallel to a binding constraint (i.e. a constraint that is satisfied as an equation at the optimal solution), the objective function will assure the same optimal value, called alternative optima.

In-Text Question

_________ are non-negative variables use in augmenting or converting an inequality sign (less or equal to) to and equation which are added to the left-hand side of the constraint.

a) Slack variables
b) Degeneracy
c) Alternative Optima
d) Algorithm

In-Text Answer

(A)

1.2 Principal Component of a Programming Model

There are three principal elements of Programming Model (i.e. decision problem). They are as follow:

a. Definition of the decision variables of the problem.

b. Determination of objective of the study, whether to minimize cost or maximize profit.

c. Specification of the limitations (constraints) under which the modeled system operates (i.e. the recognition of the limitation, requirement, bound and equality of the system).

1.3 Phases of Operations Research (OR)

Problem definition: This involves defining the scope of the problem under investigation with aim of identifying the three principal elements of the decision problem; decision variables, objective function and specification of the constraints.

Model Construction: This entails the translation of the problem definition into mathematical relationships i.e. if you want to express the decision variables mathematically in terms of its constraint and objective function.

Model Solution: This is achieved by using ll-defined optimization techniques (e.g. linear, non-linear, integer programming) to yield an optimal solution.

Model validity: It entails checking whether or not the proposed model does what it is supposed to do that is, does the model predict adequately the behaviour of the system under study.

Implementation: This involves the translation of the results into operating instructions issued in understandable form to the individual who will administer the recommended system.

1.4 Techniques use in Operations research

The following cited techniques are but a partial list of the large number of available OR tools.

Linear programming: These are Models with strict linear objective and constraint functions. It is the most prominent OR technique.

Integer programming: These are models in which the variables assume integer values.

Dynamic programming: Models in which the original model can be decomposed into smaller sub-problems.

Network programming: Models in which the problem can be modeled as a network.

Nonlinear programming: These are models are models in which the functions of the model are non-linear.

Summary for Study Session 1

In this Study Session, you have learnt that

1. **Operations research** is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defense.

Self-Assessment Question for Study Session 1

1a. What do you understand by Operations research (OR)?

b. Discuss five techniques you know that are used in Operations research.

2. Define the following Concepts in OR

a) Unbounded solution
b) Infeasible solution
c) Alternative Optima
d) Degeneracy
e) Algorithms.

3a. Distinguish between the pairs of the following terms:

i. Slack and Surplus variables

ii. Basic and non-basic variables

iii. Feasible and optimal solutions

4. Explain the Five principal phases of Operation Research.

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Study Session Two: Linear Programming I

Expected duration: 1 week or 2 contact hours

Introduction

The most common technique used in Operations Research is linear programming (LP). It applies to optimization models in which the objective and constraint functions are strictly linear. The technique is used in a wide range of applications, including agriculture, industry, transportation, economics, health systems, behavioral science, social sciences, and military shall start with the case of two-variable model and presents its graphical solution.

To illustrate the use of LP in the real world, applications are formulated and then solved in both urban and rural areas.

Learning Outcomes for Study Session 2

At the end of this Study Session, you should be able to:

2.1 Define linear programming and list some of its characteristics as a tool in OR;

2.2 Formulate correctly a linear programming problem

2.1 Definition of Linear Programming (LP)

Linear programming is an Operations Research technique which helps in making optimal decisions in this direction. LP is one of the basic tools used to allocate limited resources in order to achieve optimality or utility maximization. Hover, achieving this objective is subject to series of constraints which impose limitations.

Characteristics of a Linear Programming

There are many characteristics of a linear programming such as single objectivity, linearity, simplicity and non-negativity, but some are the basic, such as Additivity, divisibility and proportionality.

❖ **Additivity:** This means that the total value of the objective function as ll as the constraint function is equal to the sum of the individual contributions of the decision variables.

❖ **Proportionality:** It implies that the contribution of each variable in the objective function affects the level or value of the variable i.e. if the value of a variable is multiplied by a constraint, then its contribution to the objective and constraint is also multiplied by the same constraint.

❖ **Divisibility:** The means that the decision variable can theoretically assume any value including fractions in some intervals between - ∞ to + ∞.

2.2 Formulation of linear programming problems

Problem formulation is the process of transforming the verbal description of a decision problem into a mathematical form that could then be solved either manually or by employing the service of a computer system. Problem formulation is an art in itself and there are simple rules to follow.

In solving real world problems, several approximations may have to be made before a problem can be modeled mathematically. Only with adequate practice can learn how to formulate problems correctly. In general, hover, the following are steps involved.

In-Text Question

Linear programming is one of the basic tools used to allocate limited resources in order to achieve optimality or utility maximization. True or false

In-Text Answer

True

Identification of the decision variables:

These are the variables (quantities) whose values you wish to determine. The variables are closely linked to the objectives and activities of the problem. In a given problem, there could be different set of variables at different times.

Optimization of the objective function:

The decision maker is the one who specifies the overall objective of the problem. For a linear programming model, for example, the objective function might take a form of maximization profits or minimizing cost Linear Programming allows for the optimization of only one objective function. So, the decision maker must indicate a single specific objective function to be optimized.

Specification of the Constraints:

Here all stipulation requirements, restrictions and limitations are identified. These are also expressed mathematically. Indeed, there are different types of constraint use in

Operations Research depending on which type of technique is involved. But basically, for linear programming, have four major types which are as follow;

Availability or Resource Constraint:

These constraints usually reflect a limited availability of resource. It often use the words like at most, available, limited to such constraint is of the form etc

$$a_{11}x_1 + a_{22}x_2 + ... + a_{1n}x_n \leq b_1 \text{, where } a_{11}, a_{12} ... a_{1n} \text{ and } b_1 \text{ are constant.}$$

Requirement Constraint: These constraints usually reflect and imposed requirement on a problem. It uses words like at least, greater than, at minimum etc. such constraint is of the form

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \geq b_1$$

Structural or equality constraint: These are constraints that usually reflect a structural or technological relationship among variables. e.g.

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

Bound Constraints: These are constraints with only one variable i.e. non-negative ($X_i \geq 0$), a non-positive ($X_i \leq 0$) or a variable lying between a given upper or lor bound e.g. $-10 \leq X_i \leq 20$. It may even be possible that some variables are unrestricted i.e. they can assume any value (+ve, -ve or zero)

Examples of Linear Programming Problem:

Example 1

Product mixed problem

The Tor Aluminum Plc wishes to schedule the production of a kitchen appliance which requires two resources (Labour & material) the company is considering 3 different models and its production engineering ... has furnished the following data:

	Model		
	A	B	C
Labour (hrs/unit)	7	3	6
Material (kg/unit)	4	4	5
Profit (N/unit)	40	20	30

Total Labour 150 Hrs and raw-material 200kg

The supply of raw-material is restricted to 200kg per day. The daily average of man por is 150 hrs. Formulate the Linear Programming Model to determine the daily production rate of the various models in order to maximize the total profit.

Solution

(1) Identify the variables: The unknown active to be determine are the daily rate of production for the 3 models. Let us represent them as follows:

X_A ⇒ Daily production rate of model A

X_B ⇒ Daily production rate of model B

X_C ⇒ Daily production rate of model C

(2) Identify the objective function: The objective is to maximize the total profit from sales. Assuming a perfect market exist for the products such that all that is produced can be sold, the total profit from the sales becomes (To get the tot profit, quantity produced x unit profit for each A, B, C and then add together).

Max (Total profit) $Z = 40X_A + 20X_B + 30X_C$

(3) Identification of the constraints: Here have only 2 constraints which are labour and materials and the 2 are availability constraints

i. $7X_A + 3X_B + 6X_C$ ... 150(Hrs) - Labour

ii. $4X_A + 4X_B + 5X_C$ ... 200kg per day - material

iii. X_A, X_B, X_C ... 0 Bound Constraints

Therefore, the Linear Programming Model for this prob. is:

Maximize, $Z = 40X_A + 20X_B + 30X_C$

Subject to: $7X_A + 3X_B + 6X_C$ ... 150

$4X_A + 4X_B + 5X_B$ ... 200

Example 2

Reddy Mikks produces both interior and exterior paints from two raw materials, M_1 and M_2. The following table provides the basic data of the problem:

	Tons of material per tons of		
	Exterior paint	Interior paint	Max. daily
Availability (tons)			
Raw material, M1	6	4	24
Raw material, M2	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Also, the maximum daily demand of interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

Solution:

The Linear Programming has three basic components

1. Decision variables that seek to determine. For the problem at hand, need to determine the amounts to be produced of exterior and interior paints. Thus the variables are defined as:

X_1 = Tons produced daily of exterior paint

X_2 = Tons produced daily of interior paint

2. Objective (goal) that aim to optimize

To construct the objective function, the company wants to increase its profit as much as possible. Letting z represent the total daily profit (n thousands of dollars), the objective of the company is expressed as

$$\text{Maximize } z = 5x_1 + 4x_2$$

3. Constraints that you need to satisfy

The raw material restrictions are expressed verbally as

$$\begin{pmatrix} \text{Usage of raw material} \\ \text{by both paints} \end{pmatrix} \leq \begin{pmatrix} \text{Maximum raw material} \\ \text{availability} \end{pmatrix}$$

From the data of the problem, have:

Usage of raw materials M1 per day = $5x_1 + 4x_2$ tons

Usage of raw materials M2 per day = $1x_1 + 2x_2$ tons

Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons, respectively, the associated restrictions are given as

$6x_1 + 4x_2 \leq 24$ (Raw material M1)

$1x_1 + 2x_2 \leq 6$ (Raw material M2)

Demand Restrictions:

First – The difference between the daily production of interior and exterior paints does not exceed 1 ton. This translates to: $X_2 - x_1 \leq 1$

Second – Maximum daily demand of interior paint is limited to 2 tons. This translates to $x_2 \leq 2$

Finally, the non-negativity restrictions require that the variables x_1 and x_2 cannot assume negative values. This translates to

$X_1, x_2 \geq 0$

The Complete Reddy Mikks model is

Maximize z = 5x1 + 4x2

Subject to

$$6x_1 + 4x_2 \leq 24$$

$$1x_1 + 2x_2 \leq 6$$

$$-x_1 + x_2 \leq 1$$

$$x_2 \leq 2$$

$$X_1, X_2 \geq 0$$

Solution of LP Problems

There are 2 major methods of solution to solve a LP problem. These are (i) graphical solution and (ii) Simplex Method solution (Algebraic solution) and are subdivided into two: Tableau and Matrix

Graphical LP solution is only suitable for a two variable model but models with 3 or more variable the Graphical method is either impracticable or impossible.

The first step in graphical method is to plot the feasible solution or solution space which satisfies all the constraints simultaneously. The negativity restriction confirms all the feasible value the first quadrant. The space enclosed by the remaining constraint is determined by first replacing the inequalities by equality for each constraint.

Thus, yielding a straight line equation, each straight line is then on the plane, and the region in which each constraints holds in the inequality is activated is indicated by the direction of the arrow on the associated straight line.

Each point within or on the boundary of the solution space satisfies all the constraints and hence rep. a feasible point. Although, there is infinity of feasible in the solution space the optimum solution cum can be found at the corner points.

Example 3

Max $Ƶ = 5x_1 + 6x_2$

S.t $3X_1 + 2X_2 \leq 120$

$4X1 + 6X_2 \leq 260$

$X_1, X_2 \geq 0$

Solution to the problem:

for constraint (1)

$3X_1 + 2X_2 = 120$

When $X_1 = 0$, $X_2 = 60 \Rightarrow (0,60)$

When $X_2 = 0$, $X_1 = 60 \Rightarrow (40, 0)$

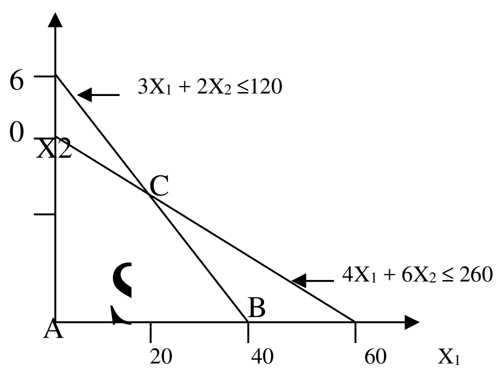
For constraint (2)

$4x_1 + 6X_2 = 260$

When X1 = 0, X2 = $^{130}/_3$ or 43.2 $\Rightarrow$ (0, 43.3)

When X2 = 0, X1 = 65 $\Rightarrow$ (65,0)

Sketching this, on a plane (graph), obtain;



The feasible solution space is regions with ‘S’.

To obtain the coordinate of point C, that is, the point of intersection, you need to solve the two equations simultaneously (If not using a graph sheet) to have $X_1 = 20$ and $X_2 = 30$.

Corner point	(X_1, X_2)	Objective function (Ƶ)
A	(0,0)	0 [5(0)+ 6(0)
B	(40,0)	200 [5(40) + 6(0)
C	(20,30)	**280** [5(20) + 6(30)
D	(0, $^{130}/_3$)	260 [5(0) + 6 ($^{130}/_3$)

There, the maximum value of Z is 280 at point (20, 30) i.e. to maximize the contribution, 20 of first product and 30 of second product should be produced by the firm.

Example 4

Min C = $3X_1 + 2X_2$

S.t: $X_1 + X_2 \geq 70$

$X1 + 2X_2 \geq 150$

$2X_1 + X_2 \geq 120$

$X1, X_2 \geq 0$

Solution

Set the linear inequalities to linear equations

Constraint (1): $X_1 + X_2 = 70$

When $X_1 =0$, X_2 70 $\Rightarrow$ (0, 70)

When $X_2 = 0$, $X_1 = 70 \Rightarrow$ (70, 0)

Constraint (2): $X_1 + 2X_2 = 150$

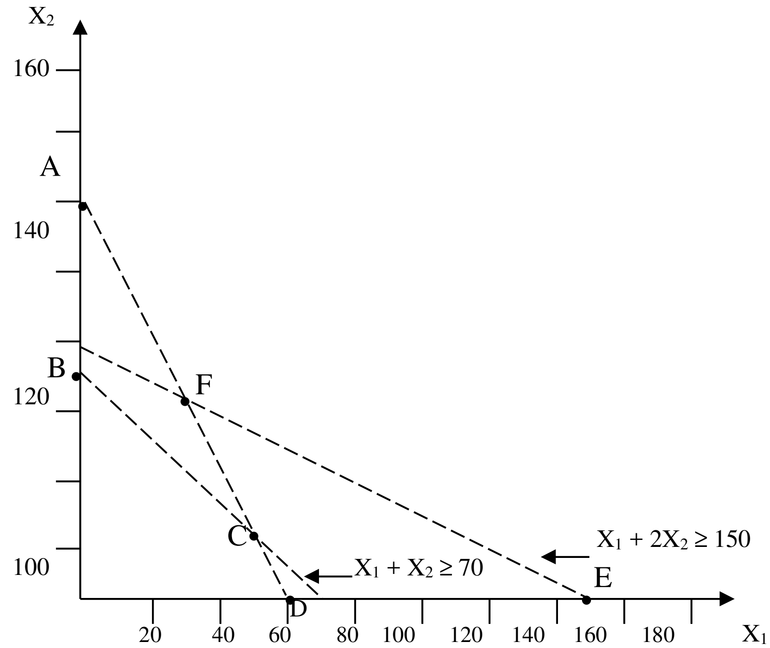
When $X_1 =0$, $X_2 = 75 \Rightarrow$ (0, 75)

When $X_2 = 0$, $X_1 = 150 \Rightarrow$ (150,0)

Constraint (3): $2X_1 + X_2 = 120$

When $X_1 = 0$, $X_2 = 120 \Rightarrow$ (0,120)

When $X_2 =0$, $X_1 = 60 \Rightarrow$ (60,0)



Corner point	**(X_1, X_2)**	**Objective function (C)**
A	(0,120)	240 [3(0)+ 2(120)]
B	(0,70)	**140** [3(0) + 2 (70)]
C	(50,20)	190 [3(50) + 2 (20)]
D	(60,0)	180 [3(60) + 2(0)]
E	(150,0)	450 [3(150) + 2(0)]
F	(30,60)	210 [3 (30) + 2 (60]]

Therefore the minimum cost is 140 at point B(0,70) i.e. to minimize the cost (C), only 70 of the second product (X_2) should be produced by the firm.

Advantages of graphical solution of LP Problems

i. It is simpler and faster with the two variable L.P problem

ii. It is mostly applicable for the two variable problems

iii. It can be read from the plane (graph) easily.

Disadvantages of graphical solution of LP Problems

(i) It is only limited to two- variable problems (models)

(ii) When there are more constraints in these two variables model, it may be difficult to solve the problem.

Summary of Study Session 2

In this Study Session, you have learnt that;

1. Linear programming is a technique in operation research to allocate scarce resources (labour, materials, machines & capital) in the best possible manner so as to minimize the cost of production or maximize the profits to be made by a firm. In linear programming models, there are three basic components namely, decision variables, constraints and the objective goal.

2. The four types of constraints in linear programming model are limited availability, requirement, structural and bound constraints. The optimum linear programming solution (under graphical method is associated with the corner points of the solution space). The graphical solution is only limited to two variables model.

Self-Assessment Question for Study Session 2

1. For the Reddy Mikks model, construct each of the following constraints and express them with a constant right-hand side.
    a. The daily demand for interior paint exceeds that of exterior paint by at least 1 ton.
    b. The daily usage of raw material M2 is at most 6 tons and at least 3 tons.
    c. The demand for interior paint cannot be less than the demand for exterior paint.
    d. The minimum quantity that should be produced for both the interior and the exterior paints is 3 tons.

2. Determine the best feasible solution among the following (feasible and infeasible) solutions of the Reddy Mikks model:
    (a) $x_1 = 1, x_2 = 4$
    (b) $x_1 = 2, x_2 = 2$
    (c) $x_1 = 3, x_2 = 1.5$
    (d) $x_1 = 2, x_2 = 1$

(e) $x_1 = 2, x_2 = -1$

3. For the feasible solution $x_1 = 2, x_2 = 2$ of the Reddy Mikks model, determine
   (a) The unused amount of raw material M1.
   (b) The unused amount of raw material M2

4. John must work at least 20 hours a ek to supplement his income while attending school. He has the opportunity to work in two retail stores. In store 1, he can work between 5 and 12 hours a ek, and in store 2 he is allod between 6 and 10 hours. Both stores pay the same hourly wage. In deciding how many hours to work in each store, John wants to base his decision on work stress. Based on interviews with present employees, John estimates that, on an ascending scale of 1 to 10, the stress factors are 8 and 6 at stores 1 and 2, respectively. Because stress mounts by the hour, he assumes that the total stress for each store at the end of the ek is proportional to the number of hours he works in the store. How many hours should John work in each store?
5. OilCo is building a refinery to produce four products: diesel, gasoline, lubricants, and jet fuel. The minimum demand (in bbl/day) for each of these products is 14,000, 30,000, 10,000, and 8,000, respectively. Iran and Dubai are under contract to ship crude to OilCo. Because of the production quotas specified by OPEC (Organization of Petroleum Exporting Countries) the new refinery can receive at least 40% of its crude from Iran and the remaining amount from Dubai. OilCo predicts that the demand and crude oil quotas will remain steady over the next ten years.
6. An industrial recycling center uses two scrap aluminum metals, *A* and *B*, to produce a special alloy. Scrap *A* contains 6% aluminum, 3% silicon, and 4% carbon. Scrap *B* has 3% aluminum, 6% silicon, and 3% carbon. The costs per ton for scraps *A* and *B* are \$100 and \$80, respectively. The specifications of the special alloy require that (1) the aluminum content must be at least 3% and at most 6%, (2) the silicon content must lie between 3% and 5%, and (3) the carbon content must be between 3% and 7%. Determine the optimum mix of the scraps that should be used in producing 1000 tons of the alloy.
7. Given the following problem,

$$
\begin{aligned}
\text{maximize } \not{Z} = \quad & 5X_1 + 4X_2 \\
\text{S.t} \quad & 6X= +4X_2 \leq 24 \\
& X_1 + 2X_2 \leq 6 \\
& -X_1 + X_2 \leq 1 \\
& X_2 \leq 2 \\
& X_1, X_2 \geq 0
\end{aligned}
$$

Solve the above problem, using graphical solution?

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Study Session Three: Linear Programming II

Expected duration: 1 week or 2 contact hours

Introduction

The transition from the geometric corner-point solution to the algebraic method entails a computational procedure that determines the corner points algebraically. This is accomplished by first converting all the inequality constraints into equations and then manipulating the resulting equations in a systematic manner. A main feature of the algebraic method is that it solves the LP in *iterations*. Each iteration moves the solution to a new corner point that has the potential to improve the value of the objective function. The process ends when no further improvements can be realized.

Learning Outcomes for Study Session 3

At the end of this Study Session, you should be able to:

3.1 Explain simplex (algorithm) solution of an LP problem.
3.2 Discuss the concept of standard form of an LP model.
3.3 Differentiate between the slack and surplus variables.
3.4 Distinguish between the basic and non-basic variables.
3.5 Explain the artificial starting solution.

3.1 Equation (Standard) Form in operations research

For the sake of standardization, the algebraic representation of the LP solution space is made under two conditions:

a. All the constraints (with the exception of the nonnegative restrictions) are equations with a nonnegative right-hand side.

b. All the variables are nonnegative.

Converting Inequality into Equations (Standard Forms)

- To convert a ($\leq$) inequality to an equation, a nonnegative **slack variable** is added to the left-hand side of the constraint.

Example
Using the constraint associated with the raw material M1 in Reddy Mikks model:

$$6x_1 + 4x_2 \leq 24$$

Defining s_1 as the slack or unused amount of M1, the constraint can be converted to the following equation.

$$6x_1 + 4x_2 + s_1 = 24, s_1 \geq 0$$

- To convert a ($\geq$) inequality to an equation, subtract a nonnegative **surplus variable** from the left-hand side of the inequality.

Example
In the diet model, the constraint representing the minimum feed requirements is given as

$$x_1 + x_2 \geq 800$$

Defining S_1 as the surplus variable, the constraint can be converted to the following equation: $x_1 + x_2 – S_1 \geq 800, S_1 \geq 0$

Converting Right-hand side to be Nonnegative

- The condition can always be satisfied by multiplying both sides of the resulting equation by -1, where necessary.

Example
The constraint:

$$-x_1 + x_2 \leq -3$$

Basic and Non-basic Variable
In the simplex method, the solution space is represented by *m* simultaneous linear equations and *n* nonnegative variables. The number of equations m is always *less than* or *equal to* the number of variables n.

- If m = n, and the equations are consistent, the system has only one solution.
- If m < n (which represents the majority of LPs), and the equations are consistent, the system of equations will yield infinity of solutions.

Having shown how the LP solution space is represented algebraically, the candidates for the optimum (i.e., corner points) are determined from the simultaneous linear equations in the following manner:

Theory:
In a set of *m by n* equations (*m<n*), if set *n-m* variables equal to zero and then solve the *m* equations for the remaining *m* variables, the resulting solution, if unique, must correspond to a corner point of the solution space.

To make a complete transition to the algebraic solution, need to refer to the corner points by their algebraic names. Specifically, the n-m variables that are set to zero are known as ***non-basic variables***. If the remaining m variables have a unique solution then they are called ***basic variables*** and their solution (obtained by solving the m equations) is referred to as the ***basic solution***.

3.2 Computational Details of the Simplex Algorithm

This section provides the computational details of a simplex iteration that include the rules for determining the entering and leaving variables as ll as for stopping the computations when the optimum solution has been reached. Use the Reddy Mikks model to explain the details of the simplex method.

The problem is expressed in standard form as

Example 1
Maximize $z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$
Subject to:

$$6x_1 + 4x_2 + s_1 = 24 \text{ (Raw materials } M1)$$
$$x_1 + 2x_2 + s_2 = 6 \text{ (Raw materials } M2)$$
$$-x_1 + x_2 + s_3 = 1 \text{ (Market limit)}$$
$$x_2 + s_4 = 2 \text{ (Demand limit)}$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

The variables s_1, s_2, s_3, and s_4 are the slacks associated with the respective constraints.
Next, express the objective equation as

$$z - 5x_1 - 4x_2 = 0$$

In this manner, the starting simplex tableau can be represented as follows:

Basic		x_1	x_2	s_1	s_2	s_3	s_4	Solution	
Z	1	-5	-4	0	0	0	0	0	z-row
s_1	0	6	4	1	0	0	0	24	s_1-row
s_2	0	1	2	0	1	0	0	6	s_2-row
s_3	0	-1	1	0	0	1	0	1	s_3-row
s_4	0	0	1	0	0	0	1	2	s_4-row

The tableau defines the current corner point by specifying its basic variables and their values, as ll as the corresponding value of the objective function, z. the non-basic variables (those not listed in the Basic-column) always equal zero.

RULE 1

For maximization (minimization) problem, if there are nonbasic variables which have a negative (positive) coefficient in z-row, select one such variable with the most negative (positive) coefficient. If all non-basic variables have positive (negative) or zero coefficients in z-row, an optimal solution has been found.

RULE 2

Find the ratios of the current right hand side to the coefficients of the entering variable (ignore ratio with zero or negative numbers in the denominator), select the minimum ratio. The variable currently in the basic which corresponds to this minimum ratio will be the variable to leave the basic* (ties for the minimum ratio should be broken arbitrarily).

RULE 3

The change of basis should now be made. The pivoting element is the tableau entry which is at the intersection of the column of the incoming variable and the row of the leaving variable. (This row is called pivoting row or pivoting equation). Divide each element in the pivoting row by the value of pivoting element and the row (R*) is then written in the 2^{nd} tableau in the same row position as it was in the initial tableau.

RULE 4

To obtain the remaining rows in the new tableau it should be firstly noted that in the column of a variable in the current basis there are all zeros except for a 1 in the row corresponding of this basic variable. This condition must now be obtained for the variable which entered in Rule 1. This is obtained by adding and subtracting suitable multiples of R* from the rows of the previous tableau. When this is done; return to rule 1.

Let us now apply these rules to the problem at hand. Is the starting solution optimal? The entering variable is x_1 because it has the ***most negative*** coefficient in the (maximization) objective equation. To determine the leaving variable directly from the tableau, compute the ratios as follows:

Basic	**Entering**	**Solut**	**Ratio (or intercept)**
s_1	6	24	$x_1 = 24/6 = 4 \leftarrow$ **minimum**
s_2	1	6	$x_1 = 6/1 = 6$
s_3	-1	1	$x_1 = 1/-1 = -1$ (ignore)
s_4	0	2	$x_1 = 2/0 = \infty$ (ignore)

The minimum nonnegative ratio corresponds to basic s_1, signifying that s_1 is the leaving variable (its value is zero in the new iteration).

Now need to manipulate the equations in the last tableau so that the *Basic* column and the *solution*-column will identify the new solution. The process is the ***Gauss-Jordan row operations.***

The Gauss-Jordan computations needed to produce the new basic solution include two types.

1. Pivot row:
   New pivot row = current pivot row ÷ Pivot element
2. All other rows, including z

New row = (current row) – (its pivot column coefficient) x (New pivot row)

These computations are applied to the preceding tableau in the following manner.

1. New pivot s_1 – row = current s_1 – row **÷ 6**
2. New z-row = current z-row – (-5) x New pivot row
3. New x_1-row = current x_1-row – (-1) x New pivot row

4. New s_3-row = current s_3-row – (-1) x New pivot row
5. New s_4-row = current s_4-row - (0) x New pivot row

The new tableau corresponding to the new basic solution thus becomes:

Basic	**Z**	**x_1**	**x_2**	**s_1**	**s_2**	**s_3**	**s_4**	**Solution**
Z	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
x_1	0	1	$-\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
s_2	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
s_4	0	0	1	0	0	0	1	2

Observe that the new tableau has the same properties as the starting tableau. When set non-basic variables x_2 and s_1 to zero, the *solution*-column automatically yields the new basic solution (x_1 = 4, s_2 = 2, s_3 = 5, s_4 = 2). The corresponding new objective value is z = 20.

Now, from the last tableau:

- Determine the entering and leaving variables.
- Compute the ratios
- Determine the pivot column, pivot row and pivot element.
- Apply the Gauss-Jordan row operations to produce the next tableau as follows:

B							**s_4**	**Solut**
							0	21
							0	3
							0	$\frac{3}{2}$
							0	$\frac{5}{2}$
							1	$\frac{1}{2}$

Because none of the z-row coefficients associated with the non-basic variables s_1 and s_2 are negative, the last tableau is optimal. The optimum solution can be read from the simplex tableau in the following manner. The optimal values of the variables in the Basic-column are given in the right-hand-side solution column and can be interpreted as:

Decision varia	**Optimum valu**	**Recommendation**
x_1	3	Produce 3 tons of exterior paint daily
x_2	$\frac{3}{2}$	Produce 1.5 tons of interior paint daily
Z	21	Daily profit is $21,000

The simplex tableau offers a alth of additional information that includes:
1. The status of the resources
2. The worth per unit (dual prices) of the resources
3. All the data required to carry out sensitivity analysis on the optimal solution.

Example 2

Solve: max Ƶ $= 5x_1 + 6x_2$

S.t $3X_1 + 2X_2 \leq 120$

$4X_1 + 6X_2 \leq 260$

$X_1, X_2 \geq 0$

Using simplex method

Solution:

Objective function max Ƶ = 5X1 + 6X2 will become

Max Ƶ - $5X_1 - 6X_2 = 0$

S.t $3x_1 + 2X_2 + S_1 = 120$

$4x_1 + 6X_2 + S_2 = 260$

$X_1, X_2, S_1, S_2 \geq$ (non-negative)

Table I:

Row	Basic	Ƶ	Non-basic variable X_1 X_2	Basic variable S_1 S_2	Solution (R.H.S)
R_o	$Ƶ_1$	-1	-5 -6	0 0	0
R_1	$+S_1$	0	3 2	1 0	120
R_2	S_2	0	4 (6)	0 1	260

To determine an entry basic variable, shall enter through X2 with the largest negative on O.F.

can now determine the leaving variable by considering the least ratio i.e. the least out of 120/2 and 260/6 which is of cause 260/6, so 6 will serves as our pivot element for table I. to obtain the next table (II), divide row 2 by the pivoting element (6) to have a new row 2 called $R^1{}_2$ and for $R^1{}_0$ & R11 e obtain them by adding and subtracting multiples of $R^1{}_2$ in order to have zeros in the remaining positions in the column under X_2

$$R^1{}_0 = R_0 + 6R^1{}_2$$

$$R^1{}_1 = R_1 - 2R^1{}_2 \quad , \quad R^1{}_2 = R_2/6$$

Table II

Row	**Basic**	**Ƶ**	**X_1**	**X_2**	**S_1**	**S_2**	**Solution**
$R^1{}_o$	$Ƶ_1$	+1	-1	0	O	1	260
$R^1{}_1$	S_1	0	($^5/_3$)	0	1	- $^1/_3$	$^{100}/_3$
$R^1{}_2$	X_2	0	$^2/_3$	1	0	$^1/_6$	$^{130}/_3$

Since are left with only X1, (with negative among X), need to look for the minimum ratio i.e. the least out of ($^{100}/_3 \div ^5/_3$=20) and ($^{130}/_3 \div ^2/_3$=65) which of course, so 5/3 will serve as our pivot element.

Find the next table (Table III).

$R^{II}{}_0 = R^I{}_o + R^{II}{}_I$

$R^{II}{}_I = R^I{}_I \div {}^5/_3$

$R^{II}{}_2 = R^I{}_2 - {}^2/_3\, R^{II}{}_I$

Table III

Row	Basic	Ƶ	X_1	X_2	S_1	S_2	Solution
R^{II}_o	$Ƶ_1$	+1	0	0	3/5	$^4/_5$	280
R^{II}_1	S_1	0	1	0	3/5	$-^1/_5$	20
R^{II}_2	X_2	0	0	1	$-^2/_5$	$+ ^3/_{10}$	30

Since there are no negative values in R^{II}_o have reached the optimal solution. Then solution is found by reading off the variable in the current basic with the corresponding values in the solution column. That is, X1 = 20, X2 = 30 & Ƶ = 280 which agrees with the graphical solution.

Example 3

Max Ƶ = $3X_1 + 4X_2$

S.t $5X_1 + 3x_2 \leq 30$

$5x_1 + 11X_2 \leq 55$

$X_2 \leq 4$

$X_1, X_2 \geq 0$

Solution:

Resolving it into a standard form, have:

Table I

Row	Basic	Ƶ	X_1	X_2	S_1	S_2	S3	Solution
R_o	$Ƶ_1$	1	-3	-4	0	0	0	0
R_1	S_1	0	5	3	1	0	0	30
R_2	S_2	0	5	11	0	1	0	55
R_3	S_3	0	0	(1)	0	0	1	4

The entry variable is X2 because it has the largest negative in row (R_o). Also the leaving variables is S3 because it is that has the lost ratio (4) i.e.

Basis	Non basic variable	Objective function	Ratio
S1	3	30	$^{30}/_{3}$= 10
S2	11	55	$^{55}/_{11}$ = 5
S3	1	4	$^{4}/_{1}$=4$\Rightarrow$least ratio

$\therefore$ The pivoting element of table I is 1 (i.e. the intersection value of the entry & leaving variables). Since the pivoting element is 1, division of R_3 by 1 leaves it unuttered. If denote the row of the second tableau by R^I_0, R^I_I, R^I_2 and R^I_3 then have $R^I_3 = R_3$. Then for table II, have the following algorithms;

$R^I_o = R_o + 4R^I_3$

$R^I_I = R^I - 3R^I_3$

$R^I_{2} = R_1\text{-}11R^1_3$

Thus, obtain; Table II

ow	Basic	Z	X1	X2	S1	S2	S3	Solution
R^I_o	Z_1	1	-3	0	0	0	4	16
R^I_1	S_1	0	5	0	1	0	-3	18
R^I_2	S_2	0	5	0	0	1	-11	11
R^I_3	X_2	0	0	1	0	0	1	4

It can be observed that the current solution is not optimal since there is a negative coefficient in R^I_o under X_1. Thus, X_1 should enter the basis.

Basis	**X1**		**Solution**	**Ratio**
S_1	5		18	$^{18}/_{5}$
S_2	5		11	11/5 $\Rightarrow$minimum
X_2	0		4	4/0 NOT APPLICABLE

Hence, S_2 leaves the basis and the pivoting element (5) is circled in table II above. If the rows of the next table are labeled $R^{II}{}_o$, $R^{II}{}_I$, $R^2{}_{II}$ & $R^{II}{}_3$ then,

$$R^{II}{}_2 = R^{I}\,2/_5$$
$$R^{II}{}_o = R^{I}{}_o + 3R^{II}{}_2$$
$$R^{II}{}_I = R^{I}{}_o - 5R^{II}{}_2$$
$$R^{II}{}_3 = R^{I}{}_3$$

Table III

Row	Basic	Ƶ	X_1	X_2	S_1 S_2 S_3	Solution
$R^{II}{}_0$	Ƶ	1	0	0	0 $^{3}/_{5}$ $^{-13}/_{5}$ (8)	$^{113}/_{5}$
$R^{II}{}_I$	S_1	0	0	0	1 -1	7
$R^{II}{}_2$	X_1	0	1	0	0 $^{1}/_{5}$ $^{-11}/_{5}$	$^{11}/_{5}$
$R^{II}{}_3$	X_2	0	0	1	0 0 1	4

It can also be deduced that the current solution is not yet optimal since there is negative coefficient S_3 in $R^{II}{}_o$. Thus S_3 should enter the basis.

Basis	**X_1**	**Solution**	**Ratio**
S_1	6	7	$^{7}/_{8} \Rightarrow$ minimum
X_1	$^{-11}/_{5}$	$^{11}/_{5}$	negative (not applicable)
X_2	1	4	$^{4}/_{1}$

Thus S1 leaves the basis. The pivoting element (8) is circled in table III. If the rows of the new tableau are labeled $R^{II}{}_o$, $R^{III}{}_I$, $R^{III}{}_2$ & $R^{III}{}_3$ then $R^{II}{}_I = {R^{II}{}_I}/_{8}$, the remaining rows are

$R^{III}{}_0 = R^{II}{}_o +^{13}/_5 R^{III}{}_I$

$R^{III}{}_2 = R^{II}{}_2 +^{11}/5R^{III}{}_I$

$R^{III}{}_3 = R^{II}{}_3 - RI^{II}{}_I$

Thus, Table IV

Row	Basic	Ƶ X1 X2	S1 S2 S3	Solution
$R^{III}{}_0$	Ƶ	1 0 0	$^{13}/_{40}$ $^{11}/_{40}$ 0	$^{199}/_8$
$R^{III}{}_I$	S_1	0 0 0	$^1/_8$ $^{-1}/_8$ 1	$^7/_8$
$R^{III}{}_2$	X_1	0 1 0	$^{11}/_{40}$ $^{-3}/_{40}$ 0	$^{33}/_8$
$R^{III}{}_3$	X_2	0 0 1	$^{-1}/_8$ $^1/_8$ 0	$^{25}/_8$

Since there are no negative values in $R^{III}{}_0$, have reached the optimal solution. The solution is found by reaching off the variables in the current basis with the corresponding values in the solution column. That is, X1= $^{33}/_8$, X_2 = $^{25}/_8$, S1 = S_2 = 0, S_3 = $^7/_8$ and Ƶ = $^{199}/_8$

Artificial Starting Solution

As demonstrated in the previous examples, LPs in which all the constraints are [≤] with non-negative right-hand sides offer a convenient all-slack starting basic feasible solution. Models that involve (=) and / or (≥) constraints do not possess this property.

The procedure for starting "ill-behaved" LPs with (=) and (≥) constraints is to allow artificial variables to play the role of slacks at the first iteration, and then dispose of them legitimately at a later iteration. Two closely related methods are introduced here: The **M-method** and the **Two-phase** method.

M-Method

The M-method starts with the LP in equation form. An equation that does not have a slack (or a variable that can play the role slack) is augmented with an artificial variable, Ri to form a starting solution similar to the all-slack basic solution. Hover, because artificial are extraneous to the LP model, use a feedback mechanism in which optimization process automatically attempts to force these variables to zero level. In other words, the final solution will be as if the artificial never existed in the first place.

The desired outcome is effected by penalizing the artificial variables in the objective function. Give M, a sufficiently large positive value (mathematically, $M \to \infty$), the objective coefficient of an artificial variable represent an appropriate penalty if:

Artificial variable objective coefficient = (-M, in maximization problem M, in minimization problem).

Using this penalty, the optimization process will automatically force the artificial to zero (provided the problem has a feasible solution).

Example 4

Minimize $z = 4x_1 + x2$

Subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_1 \geq 6$$
$$x_1 + 2x_2 \leq 4$$

$x_1, x_2 \geq 0$

The first and second equations do not have variables that can play the role of slack x_4. Thus, add artificial R_1 and R_2 in the first two equations and penalize them in the objective function with $MR_1 + MR_2$. The resulting LP is given as

Minimize $z = 4x_1 + x_2 + MR_1 + MR_2$

Subject to

$$3x_1 + x_2 \quad + R_1 \quad = 3$$
$$4x_1 + 3x_2 - x_3 + \quad R_2 \quad = 6$$
$$X_1 + 2x_2 \quad +x_4 = 4$$
$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$$

In the new model, can now use R_1, R_2 and x_4 as a starting basic solution, as the following tableau demonstrates (for convenience, the z-column is eliminated because it does not change in all the iteration).

Basic	**x_1**	**x_2**	**x_3**	**R**	**R_2**	**x_4**	**Solution**
Z	-4	-1	0	-M	-M	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Before proceeding with the simplex method computations, need to make the z-row consistent with the rest of the tableau. Specifically, in the tableau, $x_1 = x_2 = x_3 = 0$, which yields the starting basic solution $R_1 = 3$, $R_2 = 6$, and $X_4 = 4$. This solution shows that the value of z must be M x 3 + M x 6 = 9M instead of 0 as shown in the right-hand side of the z-row. This inconsistency stems from the fact that R1 and R2 have nonzero coefficients (-M, -M) in the z-row (compare with the all-slack starting solution where the z-row coefficients of the slacks are zero).

Can eliminate this inconsistency by substituting out R_1 and R_2 in the z-row using the appropriate constraint equation. In particular, notice the highlighted elements (=1) in the R_1-row and R_2-row. Multiplying each of R_1-row and R_2-row by M and adding the sum to the z-row will substitute out R_1 and R_2 in the objective row that is,

New z-row = old z-row + (M x R_1-row + M x R_2-row)

The modified tableau thus becomes (verify)

Basic	**x_1**	**x_2**	**x_3**	**R_1**	**R_2**	**x_4**	**Solution**
Z	-4+7M	-1+4m	-M	0	-0	0	9M
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Notice that the new z = 9M, which is consistent now the value of the starting basic feasible solution $R_1 = 3$, $R_2 = 6$ and $x_4 = 4$. The last tableau is ready for the application of the simplex method using the optimality and the feasibility conditions earlier discussed. Because are minimizing the objective function, variable x_1 having the most positive coefficients in the z-row (-4+7M) enter the solution. The minimum ratio of the feasibility condition specifies R_1 as the leaving variable.

Once the entering and leaving variables have been determined, the new tableau can be computed by using the familiar Gauss-Jordan operations. Notice that the new z-row is determined by multiplying the pivot row by -(-4+7M) and adding the result to the current z-row

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
Z	0	$\frac{1-5M}{3}$	-M	$\frac{4-7M}{3}$	0	0	4+2M
X_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
R_2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2
X_4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3

The last tableau shows that x_2 and R_2 are the entering and leaving variables, respectively. Continuing with simplex computations, two more iterations are needed to reach the optimum: $x_1 = 2/5$, $x_2 = 9/5$, $z = 17/5$.

Note that the artificial R_1 and R_2 leave the basic solution in the first and second iterations, a result that is consistent with the concept of penalizing artificial in the objective function.

1. The use of the penalty M may force the artificial variable to zero level in the final simplex iteration if the LP does not have a feasible solution (i.e., the constraints are not consistent). In this case, the final simplex iteration will include at least one artificial variable at a positive level.
2. Theoretically, the application of the M-technique implies $M \rightarrow \infty$. Hover, using the computer, M must be finite but sufficiently large. How large is "sufficiently large" is an open question. Specifically, M should be large enough to act as a penalty. At the same time, it should not be too large to impair the accuracy of the simplex computation because of manipulating a mixture of very large and very small numbers.

Two-Phase Method

Because of the potential adverse impact of the round off error on the accuracy of the M-method where large and small coefficients are manipulated simultaneously, the two-phase method alleviates the problem by eliminating the constant M altogether. As the name suggests, the method solves the LP in two phases: Phase 1 attempts to find a starting basic feasible solution, and, if found, Phase II is inverse to solve the original problem.

Phase I: Put the problem in equation form, and add the necessary artificial variables to the constraints (exactly as in the M-method) to secure a starting basic solution. Next, find a basic solution of the resulting equations that minimizes the sum of the artificial variables. If the minimum value of the sum is positive, the LP problem has no feasible solution, which ends the process (recall that a positive artificial variable signifies that an original constraint is not satisfied). Otherwise, proceed to Phase II.

Phase II: Use the feasible solution from Phase I as a starting basic feasible solution for the original problem.

Example 5

use the same problem as above

Phase I

Minimize $r = R_1 + R_2$

Subject to

$3x_1 + x_2 \quad + R_1 = 3$

$4x_1 + 3x_2 - x_3 + \quad R_2 = 6$

$X_1 + 2x_2 \quad +x_4 = 4$

$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$

The associated tableau is given as

Basic	**x_1**	**x_2**	**x_3**	**R_1**	**R_2**	**x_4**	**Solution**
R	0	0	0	-1	-1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

As in the M-method, R_1 and R_2 are substituted out in the r-row by using the following computations:

New r-row = Old r-row + (4x R_1-row + R_2-row)

The new r-row is used to solve Phase 1 of the problem, which yields the following optimum tableau.

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
R	0	0	0	-1	-1	0	0
X_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
X_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
X_4	0	0	1	1	-1	1	1

Because minimum r = 0, phase 1 produces the basic feasible solution $x_1 = 3/5$, $x_2 = 6/5$, and $x_4 = 1$. At this point, the artificial variables have completed their mission, and can eliminate their columns altogether from the tableau and move on to phase II

Phase II

After deleting the artificial columns, the original problem written as

Minimize $z = 4x_1 + x_2$

Subject to

$$x_1 + \frac{1}{5}x_3 = \frac{3}{5}$$

$$x_3 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Essentially, Phase I is a procedure that transforms the original constraint equations in a manner that provides a starting basic feasible solution for the problem. The tableau associated with the Phase II problem is thus given as

Basic	x_1	x_2	x_3	x_4	Solution
Z	-4	-1	0	0	0
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Again, because the basic variables x1 and x2 have nonzero coefficient in the z-row, they must be substituted out using the following computations.
New z-row, they must be substituted out using the following
New z-row = old z-row + (4 x x_1-row + 1 x x_2-row)
The initial tableau of Phase II is thus given as:

Basic	**x_1**	**x_2**	**x_3**	**x_4**	**Solution**
Z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Because are minimizing, x_3 must enter the solution. Application of the simplex method will produce the optimum in one iteration.
The removal of the columns of the artificial variables at the end of phase 1 is effected only when they are all *nonbasic* (as our example illustrates). It is possible, hover, that artificial variables may remain basic but at zero level at the end of phase I. In this case such variables, by necessity, form part of phase II starting basic solution. As such the computations in Phase II must be modified to ensure that an artificial variable never becomes positive during phase II iterations. The rules for guaranteeing that a zero basic artificial variable at the end of a phase I never becomes positive during Phase II are:

1. If in the pivot column the constraint coefficient corresponding to zero basic artificial variable is positive, it will define the pivot element automatically (because it corresponds to the minimum ratio of zero), and, as desired, the artificial variables becomes nonbasic in the next iteration.

2. If the pivot-column element is zero, the next iteration will leave the artificial variables unchanged at zero level.

3. If the pivot-column element is negative, the minimum ratio will not be associated with the basic (zero) artificial variables. In this case, if the resulting minimum ratio happens to be positive, then the artificial variables will assume a positive value in the next iteration (do you see why?) and need to prevent this from happening. To do so, force the artificial variable to leave the solution anyway. Noting that the artificial variable is at zero level; its removal from the basic solution will not affect the feasibility of the remaining basic variables.

To summarize, the rule for phase II calls for forcing the artificial variable to leave the basic solution any time its constraint coefficient in the pivot column is positive or negative. As a matter of fact, this rule can be applied at end of phase I to remove zero artificial variables from the basic solution before never start with phase II.

Summary of Study Session 3

In this Study Session, continued our discussion on solutions to linear programming problems using simplex and artificial (M & Two phase) methods. These algebraic solutions used algorithms or iterations for the computations. Simplex is often with the availability constraint while the artificial could be any of the types of constraint. These solutions look for the optimal solution for at least two decision variables.

The slack variables are used in availability constraint while the surplus is in requirement. These two variables are used to convert the canonical form (LP problem) into the standard form. Gauss-Jordan row operation is a process needed to produce the new basic solution which includes the pivot row and all other rows.

Self-Assessment Question for Study Session 3

1. Use hand computation to complete the simplex iteration of the above example and obtain the optimum solution.

2. In above example, identify the starting tableau for each of the following (independent) cases, and develop the associated z-row after substituting out all the artificial variables:

a. The third constraint is $x1 + 2x2 \geq 4$

b. The second constraint is $4x1 + 3x2 \leq 6$

c. The second constraint is $4x1 + 3x2 = 6$

d. The objective function is to maximize $z = 4x1 + x2$

3. Consider the following set of constraints:

$-2x_1 + 3x_2 = 3$ (1)

$4x_1 + 5x_2 \geq 10$ (2)

$x1 + 2x_2 \leq 5$ (3)

$6x_1 + 7x_2 \leq 3$ (4)

$4x_1 + 8x_2 \geq 5$ (5)

$x_1, x_2 \geq 0$

For each of the following problems, develop the z-row after substituting out the artificial variables:

a. Maximize $z = 5x_1 + 6x_2$ subject to (1), (3), (4)

b. Maximize $z = 2x_1 - 7x_2$ subject to (1), (2), (4) and (5)

c. Minimize $z = 3x_1 + 6x_2$ subject to (3), (4), and (5)

d. Minimize $z = 4x_1 + 6x_2$ subject to (1), (2) and (5)

e. Minimize $z = 3x_1 + 2x_2$ subject to (1) and (5)

4. Consider the following set of constraints:

$$x_1 + 2x_2 - 2x_3 + 4x_4 \leq 40$$
$$2x_1 - x_2 + x_3 + 2x_4 \leq 8$$
$$4x_1 - 2x_2 + x_3 - x_4 \leq 10$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Solve the problem for each of the following objective functions

a. Maximize $z = 2x_1 + x_2 - 3x_3 + 5x_4$

b. Maximize $z = 8x_1 + 6x_2 + 3x_3 - 2x_4$

c. Maximize $z = 3x_1 - x_2 + 3x_3 + 4x_4$

d. Maximize $z = 5x_1 - 4x_2 + 6x_3 - 8x_4$

e. Minimize $z = 4x_1 + 6x_2 + 2x_3 + 4x_4$

References

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Study Session Four: Sensitivity Analysis

Expected duration: 1 week or 2 contact hours

Introduction

In LP, the parameters (input data) of the model can change within certain limits without causing the optimum solution to change. This is referred to as *sensitivity analysis*, and will be the subject matter of this section. Later, will study *post-optimal analysis* which deals with determining the new optimum solution resulting from making targeted changes in the input data.

Learning Outcomes for Study Session 4

At the end of this Study Session, you should be able to:

4.1 Discuss the concept “Sensitivity Analysis”;
4.2 Define the term “ Shadow price”;
4.3 Differentiate between Feasibility and Optimality ranges; and
4.4 Identify the effects of sensitivity of an LP problem.

4.1 Concept and Definition

In LP models, the parameters are usually not exact. With sensitivity analysis, can ascertain the impact of this uncertainty on the quality of the optimum solution. For example, for an estimated unit profit of a product, if sensitivity analysis reveals that the optimum remains the same for a $\pm$ 10% change in the unit profit, can conclude that the solution is more robust than in the case where the indifference range is only $\pm$ 1%.
The purpose of sensitivity analysis or post-optimality analysis is to study how the optimal solution from a particular model will change with changes in the input (data) coefficients.

For example in LP models, the coefficients of the objective function and the constraints are supplied as input (data) or as parameters to the model. The optimal solution obtained by the simplex method is based on the values of these coefficients. In practice, the values of these coefficients are seldom known with absolute certainty because many of these coefficients are functions of some uncontrollable parameters.

For instance, **the future demands**, **the cost of raw materials**, or **the cost of energy**, resources cannot be predicted with complete accuracy before the problem was solved. Hence the solution of a practical problem is not complete with the mere determination of the optimal solution.

In-Text Question

In Linear Programming models, the parameters are usually exact. True or false

In-Text Answer

False

Each variation in the values of the data coefficients changes the LP problem, which in turn may affect the Optimal Solution found earlier. In other to develop an overall strategy to meet the various contingencies, one has to study how the optimal solution will Age due to Ages in the input coefficients. This is known as Sensitivity analysis. Other reasons for performing a sensitivity analysis includes:

There may be some data coefficients or parameters of the LP which are controllable; e.g. availability of capital, raw materials or machine capacities. Sensitivity analysis enables one to study the effect of changing these parameters on the optimal solution. If it turns out that the Optimal value (profit/cost) changes (in our favour) by a considerable amount for a small change in the given parameter, then it may be worthwhile implementing some of these changes.

For example, if increasing the availability of labour by allowing overtime contributes to a greater increase in the maximum returns as compared to the increased cost of overtime labour, then one might want to allow overtime option.

In many cases, the values of the data coefficients are obtained by statistical estimation procedures on past figures as in the case of sales forecasts, price estimates and cost data. These estimates, in general, may not be very accurate. If can identify which of the parameters affect the objective value most, then can obtain better estimates of these parameters. These will increase the reliability of our model and the solution. The following changes in the data affect the solution.

i. Changes in the cost coefficient (the (i)
ii. Changes in the RHS constants (bi)
iii. Changes in the Constraints or coefficients matrix

(a) Adding new activities or variables
(b) Changing existing columns
(c) Adding new constraints.

Graphical Sensitivity Analysis
This section demonstrates the general idea of sensitivity analysis. Two cases will be considered:

Sensitivity of the optimum solution to changes in the availability of the resources (right-hand side of the constraints).

Sensitivity of the optimum solution to changes in unit profit or unit cost (coefficients of the objective function).

Will consider the two cases separately, using examples of two-variable graphical LPs.

Example 1 *(Changes in the Right-Hand Side)*
JOBCO produces two products on two machines. A unit of product 1 requires 2 hours on machine 1 and 1 hour on machine 2. For product 2, a unit requires 1 hour on machine 1 and 3 hours on machine 2. The revenues per unit of products 1 and 2 are \$30 and \$20, respectively. The total daily processing time available for each machine is 8 hours.

Letting x_1 and x_2 represent the daily number of units of products 1 and 2, respectively, the LP model is given as

$$\text{Maximize } z = 30x_1 + 20x_2$$

Subject to

$$2x_1 + x_2 \leq 8 \quad \text{(Machine 1)}$$
$$x_1 + 3x_2 \leq 8 \quad \text{(Machine 2)}$$
$$x_1, x_2 \geq 0$$

Figure 4.1 illustrate the change in the optimum solution when change are made in the capacity of machine 1. If the daily capacity is increased from 8 hours to 9 hours, the new optimum will occur at point G. The rate of change in optimum z resulting from changing machine 1 capacity from 8 hours to 9 hours can be computed as follows:

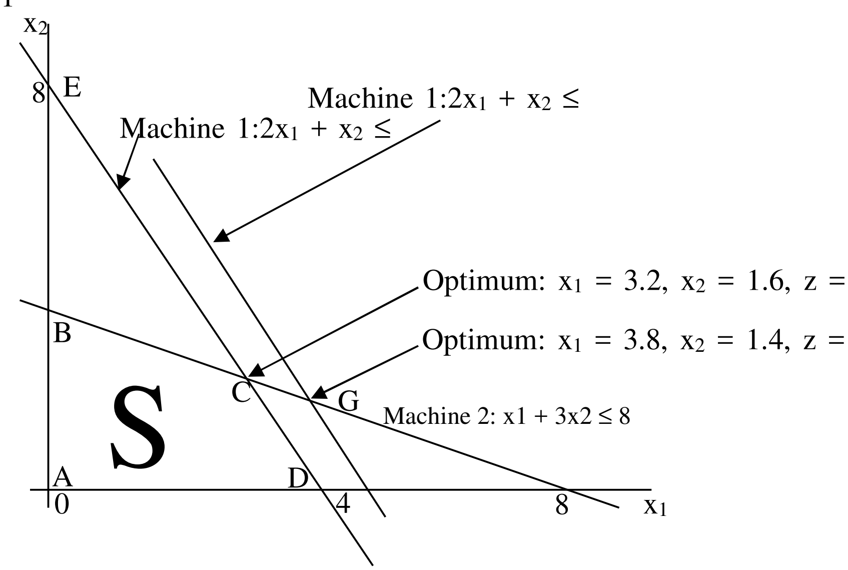
Rate of revenue change resulting from increasing machine 1 capacity by 1hr (point C to point G)

$$= \frac{zG - zC}{\text{(capacity change}} = \frac{142 - 128}{9 - 8} = \$14.00/\text{hr}$$

The computed rate provides a *direct link* between the model input (resources) and its output (total revenue) that represents the **unit worth of a resource** (in $/hr) – that is, the change in the optimal objective value per unit change in the availability of the resource (machine capacity).
This means that a unit increase (decrease) in machine 1 capacity will increase (decrease) revenue by $14.00. Although *unit worth of a resource* is an apt description of the rate of change of the objective function, the technical name **dual** or **shadow price** is now standard in the LP literature and all software packages and, hence, will be used throughout this Study Session.

Figure 4.1: Graphical sensitivity of optimal solution to changes in the availability of resources (right-hand side of the constraints)

Fig. 4.1



Looking at figure 4.1, can see that the dual price of $14.00/hr remains valid for changes (increases or decreases) in machine 1 capacity that move its constraint parallel to itself to any point on the line segment *BF*. This means that the range of applicability of the given dual price can be computed as follows:

Minimum machine 1 capacity [at *B* = (0, 2.67)] = 2 x 0 + 1 x 2.67 = 2.67hr
Minimum machine 1 capacity [at *F* = (8, 0)] = 2 x 8 + 1 x 0 = 16 hr

can thus conclude that the dual price of $14.00/hr will remain valid for the range

26.67 hrs ≤ Machine 1 capacity ≤ 16 hrs

Changes outside this range will produce a different dual price (worth per unit).

Using similar computations, you can verify that the dual price for machine 2 capacity is \$2.00/hr and it remains valid for changes (increases or decreases) that move its constraint parallel to itself to any point on the line segment *DE*, which yields the following limits:

Minimum machine 2 capacity [at D = (4, 0)] = 1 x 4 + 3 x 0 = 4 hr
Maximum machine 2 capacity [at E = (8, 0)] = 1 x 0 + 3 x 8 = 24 hr

The conclusion is that the dual price of \$2.00/hr for machine 2 will remain applicable for the range
4 hr ≤ Machine 2 capacity ≤ 24 hr

The computed limits for machine 1 and 2 are referred to as the **feasibility ranges**. All software packages provide information about the dual prices and their feasibility ranges. The dual prices allow making economic decisions about the LP problem, as the following questions demonstrate:

Example 2 (Changes in the Objective Coefficients)
Figure 3.13 shows the graphical solution space of the JOBCO problem presented in Example 4.2. The optimum occurs at point *C* (x_1 = 3.2, x2 = 1.6, *z* = 128). Changes in revenue units (i.e., objective-function coefficients) will change the slope of *z*. Hover, as can be seen from the figure, the optimum solution will remain at point *C* so long as the objective function lies between lines *BF* and *DE*, the two constraints that define the optimum point. This means that there is a range for the coefficients of the objective function that will keep the optimum solution unchanged at *C*. can write the objective function in the general format

$$\text{Maximize } z = c_1x_1 + c_2x_2$$

Imagine now that the line z is pivoted at C and that it can rotate clockwise and counterclockwise. The optimum solution will remain at point C so long as $z = c_1x_1 + c_2x_2$ lies between the two lines $x_1 + 3x_2 = 8$ and $2x_1 + x_2 = 8$. This means that the ratio c_1/c_2 can vary between 1/3 and 2/1, which yields the following condition:

$$\frac{1}{3} \leq \frac{c_1}{c_2} \leq \frac{2}{1} \; or \; .333 \leq \frac{c_1}{c_2} \leq 2$$

Figure 4.2: Graphical sensitivity of optimal solution to change in the revenue units (coefficients of the objective function)

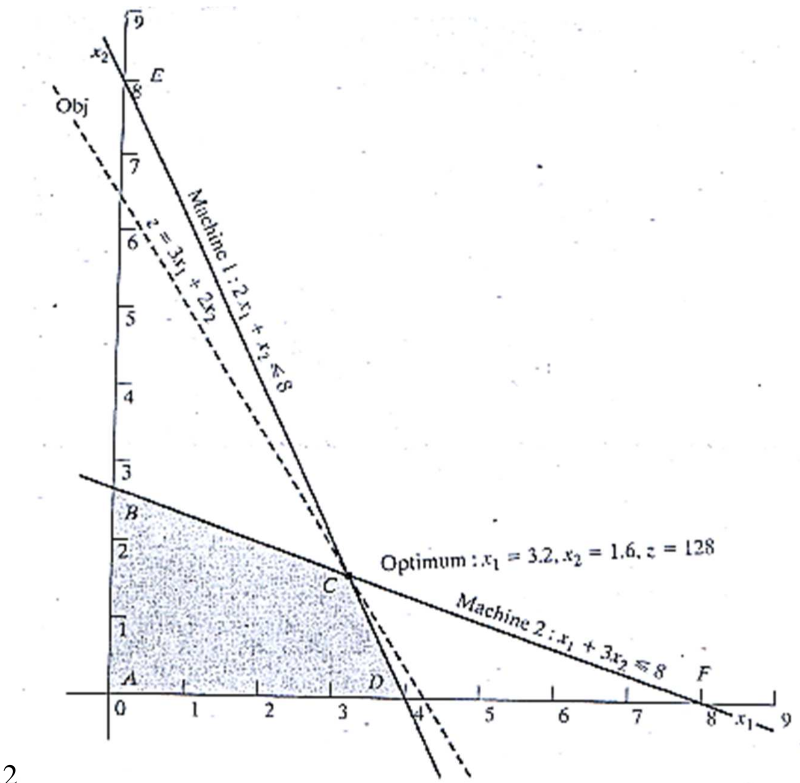


Fig. 4.2

This information can provide immediate answers regarding the optimum solution as the following questions demonstrate:

Question 1: Suppose that the unit revenues for products 1 and 2 are changed to \$35 and \$25, respectively. Will the current optimum remain the same? The new objective function is

Maximize $z = 35x_1 + 25x_2$

The solution at *C* will remain optimal because $c_1/c_2 = 35/25 = 1.4$ remains within the optimality range (.333, 2). When the ratio falls outside this range, additional calculations are needed to find the new optimum. Notice that although the values of the variables at the optimum point *C* remain unchanged, the optimum value of z change to 35 x (3.2) + 25 x (1.6) = \$152.00.

Question 2: Suppose that the unit revenue of product 2 is fixed at its current value of c_2 = \$20.00. What is the associated range for c_1, the unit revenue for product 1 that will keep the optimum unchanged?

The solution:

Substituting $c_2 = 20$ in the condition $\frac{1}{3} \leq \frac{c_1}{c_2} \leq 2$, get

$$\frac{1}{3} x 20 \leq c_1 \leq 2 x 20$$

Or

$$6.67 \leq c_1 \leq 40$$

This range is referred to as the **optimality range** for c_1, and it implicitly assumes that c_2 is fixed at $20.00.

can similarly determine the *optimality range* for c_2 by fixing the value of c_1 at $30.00. Thus

$$c_2 \leq 30 \text{ x } 3 \text{ and } c_2 \geq 30/2$$

Or

$$15 \leq c_2 \leq 90$$

As in the case of the right-hand side, all software package provide the optimality ranges. Section 3.6.4 shows how AMPL, Solver, and TORA generate these results.

Remark: Although the material in this section has dealt only with two variables, the result lay the foundation for the development of sensitivity analysis for the general LP problem above.

Summary of Study Session 4,

In this study session, you have learnt that:

Sensitivity analysis (SA) is a process which shows the behaviour or pattern of the optimum solution when the parameters (input data) of a model is changed within certain limits. Majorly, SA can be carried out by changes in either the availability of the resources (the constraints) or unit profit (unit cost) in the objective function. Dual or Shadow price is the change in the optimal objective value per unit change in the availability of the resource (machine capacity).

Feasibility and Optimality ranges are respectively as a result of the changes in the constraints (RHS) and the objective function.

Self-Assessment Question Testing Learning Outcomes for Study Session 4

1. A company produces two products, *A* and *B*. The unit revenues are \$2 and \$3, respectively. Two raw materials, M1 and M2, used in the manufacture of the two products have respective daily availabilities of 8 and 18 units. One unit of A uses 2 units of *M1* and 2 units of M2, and 1 unit of B uses 3 units of M1 and 6 units of M2.
   a. Determine the dual prices of M1 and M2 and their feasibility ranges.
   b. Suppose that 4 additional units of M1 can be acquired at the cost of 30 cents per unit. Would you recommend the additional purchase?
   c. What is the most the company should pay per unit of M2?
   d. If M2 availability is increased by 5 units, determine the associated optimum revenue.

2. Wild ST produces two types of cowboy hats. A Type 1 hat requires twice as much labour time as a Type 2. If all the available labour time is dedicated to Type 2 alone, the company can produce a total of 400 Type 2 hats a day. The respective market limits for the two types are 150 and 200 hats per day. The revenue is \$8 per Type 1 hat and \$5 per Type 2 hat.
   a. Use the graphical solution to determine the number of hats of each type that maximizes revenue.
   b. Determine the dual price of the production capacity (in terms of the Type 2 hat) and the range for which it is applicable
   c. If the daily demand limit on the Type 1 hat is decreased to 120, use the dual price to determine the corresponding effect on the optimal revenue.
   d. What is the dual price of the market share of the Type 2 hat? By how much can the market share be increased while yielding the computed worth per unit?

3. Consider Problem 1,
   a. Determine the optimality condition for $\frac{C_A}{C_B}$ that will keep the optimum unchanged.
   b. Determine the optimality ranges for C_A and C_B, assuming that the other coefficient is kept constant at its present value.
   c. If the unit revenues C_A and C_B are changes simultaneously to \$5 and \$4, respectively, determine the new optimum solution.
   d. If the changes in (c) are made one at a time, what can be said about the optimum solution?

4. In the Reddy Mikks model of Example 2.2-1;
   a. Determine the range for the ratio of the unit revenue of exterior paint to the unit revenue of interior paint.
   b. If the revenue per ton of exterior paint remains constant at $5000 per ton, determine the maximum unit revenue of interior paint that will keep the present optimum solution unchanged.
   c. If for marketing reasons the unit revenue of interior paint must be reduced to $3000, will the current optimum production mix change?

5. In Problem 2,
   a. Determine the optimality range for the unit revenue ratio of the two types of hats that will keep the current optimum unchanged.
   b. Using the information in (a), will the optimal solution change if the revenue per unit is the same for both types?

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Study Session Five: Duality Theory I

Expected duration: 1 week or 2 contact hours

Introduction

The **dual** problem is an LP defined directly and systematically from the **primal** (or original) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other.

In most LP treatments, the dual is defined for various forms of the primal depending on the sense of optimization (maximization or minimization), types of constraints ($\leq$, $\geq$ or $=$), and orientation of the variables (non negative or unrestricted). This type of treatment is somewhat confusing, and for this reason offer a *single* definition that automatically subsumes *all* forms of the primal.

Our definition of the dual problem requires expressing the primal problem in the *equation form* presented in Study Session 2 above (all the constraints are equations with nonnegative right-hand side and all the variables are nonnegative). This requirement is consistent with the format of the simplex starting tableau. Hence, any results obtained from the primal optimal solution will apply directly to the associated dual problem.

Learning Outcomes for Study Session 5

At the end of this Study Session, you should be able to:

5.1 Explain the concept "Duality theory".

5.2 Discuss the rules involve for converting a primal to a dual problem

5.3 Convert properly a problem from primal to dual and vice versa.

5.1 Construction of the Dual from the Primal variables

To show how the problem is constructed, define the primal in *equation form* as follows:

Primal problem

Maximize $z = \sum_{j=1}^{n} c_j x_j$

Subject to, $\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, \ldots, m$

$x_j \geq 0, j = 1, 2, \ldots, n$

The variables x_j, j = 1, 2, n, include the surplus, slack, and artificial variables, if any,

Dual problem

Minimize $W = \sum_{j}^{n} b_j y_j$

Subject to, $\sum_{j=1}^{n} a_{ij} y_j = c_i, i = 1, 2, \ldots, m$

$y_j \geq 0, j = 1, 2, \ldots, n$

The variables y_j, j = 1, 2, …, n, include the surplus, slack, and artificial variables, if any

5.2 Construction of the Dual from the Primal

TABLE 5: <u>Construction of the Dual from the Primal</u>

Primal variables

	x_1	x_2	…	x_j	…	x_n	
Dual variables	c_1	c_2	…	c_j	…	c_n	**Right-hand side**
y_1	a_{11}	a_{12}	…	a_{ij}	…	a_{1n}	b_1
y_2	a_{21}	a_{22}	…	a_{2j}	…	a_{2n}	b_2
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
y_m	a_{m1}						

TABLE 5.2 Rules for Constructing the Dual Problem

		Dual problem	
Primal problem objective"	*Objective*	*Constraints type*	*Variables sign*
Maximization Minimization	Minimization Maximization	≥ ≤	Unrestricted Unrestricted

All primal constraints are equations with nonnegative right-hand side and all the variables are nonnegative.

The rules for determining the sense of optimization (maximization or minimization). The type of the constraint (≤, ≥, or =), and the sign of the dual variables are summarized in Table 5.2. Note that the sense of optimization in the dual is always opposite to that of the primal. An easy way to remember the constraint type in the dual (i.e., ≤ or ≥) is that if the dual objective is *minimization* (i.e., pointing *down*), then the constraints are all of the type ≥, (i.e., pointing *up*). The opposite is true when the dual objective is maximization.

The following examples demonstrate the use of the rules in Table 4.2 and also show that our definition incorporates all forms of the primal automatically.

Example 1

Primal	Primal in equation form	Dual variables
Maximize $z = 5x_1 + 12x_2 + 4x_3$ subject to $x_1 + 2x_2 + x_3 \leq 10$ $2x_1 - x_2 + 3x_3 = 8$ $x_1, x_2, x_3 \geq 0$	Maximize $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$ subject to $x_1 + 2x_2 + x_3 + x_4 = 10$ $2x_1 - x_2 + 3x_3 + 0x_4 = 8$ $x_1, x_2, x_3, x_4 \geq 0$	y_1 y_2

Dual Problem

Minimize $w = 10y_1 + 8y_2$

subject to:

$$y_1 + 2y_2 \geq 5$$
$$2y_1 - y_2 \geq 12$$
$$y_1 + 3y_2 \geq 4$$
$$\left.\begin{array}{r} y_1 + 0y_2 \geq 0 \\ y_1, y_2 \text{ unrestricted} \end{array}\right\} \longrightarrow (y_1 \geq 0,\ y_2 \text{ unrestricted})$$

Table 5-1-1 shows how the dual problem is constructed from the primal. Effectively, have.

1. A dual variable is defined for each primal (constraint) equation.
2. A dual constraint is defined for each primal variable.
3. The constraint (column) coefficients of a primal variable define the left-hand-side coefficient of the dual constraint and its objective coefficient define the right-hand side.
4. The objective coefficients of the dual equal the right-hand side of the primal constraint equations.

Example 2

Primal	Primal in equation form	Dual variables
Maximize $z = 15x_1 + 12x_2$ subject to $x_1 + 2x_2 \geq 3$ $2x_1 - 4x_2 \leq 5$ $x_1, x_2, \geq 0$	Maximize $z = 15x_1 + 12x_2 + 0x_3 + 0x_4$ subject to $x_1 + 2x_2 + x_3 + 0x_4 = 3$ $2x_1 - 4x_2 + 0x_3 + x_4 = 5$ $x_1, x_2, x_3, x_4 \geq 0$	y_1 y_2

Dual Problem

Minimize $w = 3y_1 + 5y_2$

subject to:

$$y_1 + 2y_2 \leq 15$$
$$2y_1 - 4y_2 \leq 12$$
$$\left.\begin{array}{l} -y_1 \leq 0 \\ y_2 \leq 0 \end{array}\right\} \rightarrow (y_1 \geq 0, y_2 \leq 0)$$
$$y_1, y_2 \text{ unrestricted}$$

Example 3

Primal	Primal in equation form	Dual variables
Maximize $z = 5x_1 + 6x_2$ subject to $x_1 + 2x_2 = 5$ $-x_1 + 5x_2 \geq 3$ $4x_1, 7x_2, \leq 8$ x_1 unrestricted, $x_2 \geq 0$	Substitute $x_1 = x_1 - x_{\bar{1}}$ Maximize $z = 5x_1 + 5x_{\bar{1}} + 6x_2$ subject to $x_1 + x_{\bar{1}} + 2x_2 = 5$ $-x_1 + x_{\bar{1}} + 5x_2 - x_3 = 3$ $4x_1 - 4x_{\bar{1}}, + 7x_2 + x_4 = 8$ $x_{\bar{1}}, x_1, x_2, x_3, x_4 \geq 0$	 y_1 y_2 y_3

Dual Problem

Minimize $z = 5y_1 + 3y_2 + 8y_3$

subject to

$$\left.\begin{array}{l} y_1 - y_2 + 4y_3 \geq 5 \\ -y_1 + y_2 - 4y_3 \geq -5 \end{array}\right\} \rightarrow (y_1 - y_2 + 4y_3 = 5)$$

$$2y_1 + 5y_2 + 7y_3 \geq 6$$

$$\left.\begin{array}{l} -y_2 \geq 0 \\ y_3 \geq 0 \\ y_1, y_2 \text{ unrestricted} \end{array}\right\} \rightarrow (y_1 \text{ unrestricted}, y_2 \leq 0, y_3 \geq 0)$$

The first and second constraints are replaced by an equation. The general rule in this case is than an unrestricted primal variable always corresponds to an equality dual constraint. Conversely, a primal equation produced an unrestricted dual variable, as the first primal constraint demonstrates.

Summary of the Rules for Constructing the Dual. The general conclusion from the proceeding examples is that the variables and constraints in the primal and dual problems are defined by the rules in Table 5.3. It is a good exercise to verify that these explicit rules are subsumed by the general rules in Table 5.2.

TABLE 5.3 Rules for Constructing the Dual Problem

Maximization problem minimization problem		Minimization problem
Constraints		*Variables*
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
=	$\Leftrightarrow$	Unrestricted
Variables		*Constraints*
≥ 0	$\Leftrightarrow$	$\geq$

≤ 0	$\Leftrightarrow$	$\leq$
Unrestricted	$\Leftrightarrow$	$=$

Note that the table does not use the designation primal and dual. What matters here is the sense of optimization. If the primal is maximization, then the dual is minimization, and vice versa.

Summary of Study Session 5

In this Study Session, you have learnt that;

The primal of an LP problem is the original of that problem. Dual problem in a linear programming model is the opposite (inverse) of that problem. The number of variables in a primal determine the number of constraints in its dual. If a primal variable is restricted then its dual constraint will be structural. The dual objective type (e.g. minimization) is the opposite of the primal (minimization).

Self-Assessment Question of Study Session 5

1. In Example 5.1.1, derive the associated dual problem if the sense of optimization in the primal problem is changed to minimization.
2. In Example 5.1.2, derive the associated dual problem given at the primal problem is augmented with a third constraint, $3x_1 + x_2 = 4$.

3. In Example 5.1.3, show that even if the sense of optimization in the primal is changed to minimization, an unrestricted primal variable always corresponds to an equality dual constraint.

4. Write the dual for each of the following primal problems:

(a) Maximize $z = -5x_1 + 2x_2$

Subject to

$$-x_1 + x_2 \leq -2$$

$2x_1 + 3x_2 \leq 5$

$x_1, x_2 \geq 0$

(b) Minimize $z = 6x_1 + 3x_2$

Subject to

$$6x_1 - 3x_2\ x_3 \geq -2$$

$3x_1 + 4x_2 + x_3 \leq 5$
$x_1, x_2 \geq 0$

(c) Minimize $z = x_1 + x_2$
subject to
$2x_1 - x_2 = 5$
$3x_1 - x_2 = 6$
x_1, x_2 unrestricted

5. Consider Example 5.1.1. The application of the simplex method to the primal requires the use of an artificial variable in the second constraint of the standard primal to secure a starting basic solution. Show that the presence of an artificial primal in equation form variable does not affect the definition of the dual because it leads to a redundant dual constraints.

6. Choose **True or False** for each of the following statement.
   a. The dual of the dual problem yields the original primal.
   b. If the primal constraint is originally in equation form, the corresponding dual variable is necessarily unrestricted.
   c. If the primal constraint is of the type ≤, the corresponding dual variable will non-negative (non-positive) if the primal objective is maximization (minimization).
   d. If the primal constraint is of the type ≥, the correspond dual variable will be non-negative (non-positive) if the primal objective is minimization (maximization).
   e. An unrestricted primal variable will result in an equality dual constraint.

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Study Session Six: Transportation Problem

Expected duration: 1 week or 2 contact hours

Introduction

Transportation is a special linear programming. While linear programming is centered on minimizing cost of production or maximizing profit from sales, transportation on the other hand is centered on minimizing the cost of moving a given quantity of an item from specified sources to specified locations. This Study Session is devoted to scientific methods of solving complex transportation problems. It involves the movement of specified quantity of an item from m sources to n locations at minimum cost.

Learning Outcomes for Study session 6

At the end of this Study Session, you should be able to:

6.1 Explain the concept " transportation in an LP problem".
6.2 State clearly the assumptions of a transportation model.
6.3 Discuss the rules involve in the method solving a transportation problem.
6.4 Illustrate with detailed examples the solution method in a transportation problem.

6.1 The Concept "Transportation in an LP Problem".

In order to simplify our calculations, we shall define the following terms which are indicated in table 6.1 (below):

Table 6.1

C11	….				C1m	a_1
						a_2
						a_i
.cn1	……				Cnm	
b_1	b_2	b_3				b_m Σb_j

m= number of sources, n = number of locations, L_{ij} = cell in source i and location j C_{ij} = unit cost of item from source i to location j, a_i = quantity of item available in source I b_j = quantity of the item demanded in location j m

Σa_i = total item available in all m sources,

n

Σb_j= total item demanded in all n locations, Moreover it is required that:

m n

$\Sigma a_i = \Sigma b_j$ ……..……………………… *Balanced Transportation Problem.*

6.2 Methodology of Solving Transportation Problems

The solution to transportation problem is in two phases. The first phase will produce the initial feasible solution. Three methods will be discussed in succession in this phase.

They are:

i. North ST corner method
ii. Least cost method
iii. Vogel's approximation method

It is expected that method (ii) will improve on method (i) while method (iii) will improve on method (ii)

Phase two of the solution will produce the optimum solution to the transportation problem. The optimization technique is applicable to any of the initial feasible solutions obtained in phase one. For this method to be applicable, it is required that the number of cells where items are allocated in the initial feasible solution in phase one is equal to m + n – 1.

Illustrative Data

In order to drive home our points, the data below will be used to explain our methods.

	Lagos	Ibadan	Enugu	Supply
Abuja	800	700	600	400
Sokoto	1600	100	900	1200
Kano	1900	1800	1200	900
Demand	1400	950	150	

Table 6.2

The above table shows the supply of rams by a cattle dealer from Abuja, Sokoto and Kano and the demand for the rams in Lagos, Ibadan and Enugu. The figures in the small boxes in the cells are the costs in Naira of transporting one ram from source i to location j.

6.2.1 The North st corner Method

In this method, cell-to-cell allocation of item starts from the North - st corner of the table. Maximum number of item, based on the supply and demand is allocated from cell to cell until all supplies are exhausted and all demands are satisfied. The total transportation cost is obtained by summing up the products of the number of the item allocated to cells and their respective unit costs.

Example

Use the North st - corner method to allocate all rams supplied to table 2.2 cells where they are demanded. Hence or otherwise, calculate the total cost (C) of transportation.

Solution: Allocation

Starting from the North - st corner:

1. Allocate 400 rams to cell L_{11} to exhaust the supply from Abuja leaving demand 1000 rams in Lagos. Cross out satisfied row 1.
2. Allocate 1000 rams to cell L_{21} to satisfy demand in Lagos, leaving supply 200 rams in Sokoto. Cross out satisfied column 1.
3. Allocate 200 rams to cell L_{22} to exhaust supply from Sokoto, leaving demand 750 rams in Ibadan. Cross out satisfied row 2.
4. Allocate 750 rams to cell L_{32} to satisfy the demand in 1 Ibadan, leaving supply 150 rams in Kano.
5. Allocate 150 rams to cell L_{33} to exhaust the supply in 10 and satisfy the demand in Enugu. This complete the allocation.

Observe that m + n – l = 3 + 3 – l = 5 = number of cells with allocation.

The associated transportation cost is:

C = Σ cell allocation x unit cost

= 400 x 800 + 1000 x 1600 + 200 x 1000 + 750 x 1800 + 150 x 1200

i.e C = 320,000 + 1,600,000 + 200,000 + 1,350,000 + 180,000

= ₦3.650.000

6.2.2 The Least Cost Method

In this method, the cell with the least unit cost is first id for allocation. For any cell considered with this method, allocation is done the same way as in the comer method. Moreover, allocation is not i cross out cells.

Example

Solve the transportation problem for the given data. Using the least cost method.

Solution:

	Lagos	**Ibadan**	**Enugu**	**Supply**
Abuja	800	700 250	600 150	~~400~~ 250
Sokoto	1600 500	1000 700	900	~~1200~~ 500
Kano	1900 900	1800	120	900
Demand	1400 900	950 700	150	

Table 6.4

Solution: Allocation

1. Allocate 150 rams to cell L_{13}, with the least unit cost N600, to satisfy the demand at Enugu, leaving supply 250 rams in Abuja. Cross out satisfied column 3.
2. Allocate 250 rams to cell L_{12}, with the next least unit cost N700, to exhaust the supply from Abuja, leaving demand 700 rams at Ibadan. Cross out exhausted row 1.
3. Allocate 700 rams to cell L_{22}. With the next least unit cost N1, 000 to satisfy the demand at Ibadan, leaving supply 500 rams at Sokoto. Cross out satisfied column 2.

4. Allocate 500 rams to cell L_{21}. With the next least unit cost N1600 to exhaust the supply from Sokoto, leaving 900 rams demand at Lagos. Cross out exhausted row 2.
5. Allocate 900 rams to cell L_{31}, the only uncrossed cell, to satisfy the demand in Lagos and exhaust the supply from Kano. This complete the allocation. Observe that m + n – 1 = 3 + 3 – 1 = 5 = number of cells with allocation.

	1	2	3	Sources
A	2t	t-10	90	45
B	70	50	2t-20	40
C	t+2	85	t+1	35
Location	20	60	40	

The associated transportation cost is

C = 250 x 700 + 150 x 600 + 500 x 1600 + 700 x 1000 + 900 x 1900

= 175,000 + 900,000 + 800,000 + 700,000 + 1,710,000

= N3,475,000

This is an improvement on the North st corner method.

Example

Table 6.1.2a below shows bags of cement in sources A, B and C waiting to be transported to locations 1, 2 and 3.

Table 6.1.2a

The figures in the small boxes in the cells are unit costs in Naira of transporting a bag of cement from source i to location j (a) If the total cost of transporting all bags of cement from their sources to their respective locations via the North st corner method is N7250, find the value of t. (b) Hence or otherwise, determine the total transportation cost if the least cost method is adopted.

Solution

1	2	3	Source
2t 20	t-10 25	90	~~45~~ 25
70	50 35	2t-20 5	~~40~~ 5
t+25	85	t+10 35	~~35~~
20	60 35	40 35	

a.

Table 6.5b

Location

The allocations via N.W corner method are as shown in table 6.5b above

$\therefore$ Total transportation cost, via the North st Corner method is:

$20 \times 2t + 25(t - 10) + 35 \times 50 + 5(2t - 20) + 35(t + 10)$

$= 40t + 25t - 250 + 1750 + 10t - 100 + 35t + 350$

$= 110t + 1750 = 7250$ (given)

$\Rightarrow 110t = 5500$

$\Rightarrow t = 50$

b. Substituting t = 50 into the small boxes to obtain the table 6.5c below.

	1	2	3	Source
A	[100] 20	[40] 25	[90]	~~45~~ 25 /
B	[70]	[50] 35	[80] 5	~~40~~ 5 /
C	[75]	[85]	[60] 35	~~35~~
Locations	20 /	60 / 35 /	40 / 35 /	

Table 6.1.2a

The allocation via the least cost method is as made in table 6.5c above. The corresponding total transportation cost is

C = 45 x 40 + 20 x 70 + 15 x 50 + 5 x 80 + 35 x 60

= 1800 + 1400 + 750 + 400 + 2100

= ₦6450

6.2.3 Vogel's approximation method

This method is designed to improve on the least cost and the North-st corner methods. The steps involved are explained as follows:

1. Construct (he row and the column penalties for the unit costs. The penalty for each row/column is constructed by subtracting the smallest unit cost from the next smallest unit cost in the row/column.
2. Select for allocation, the cell with the least cost in the highest row/column penalty. When there is a tie in row/column penalty, select row or column arbitrarily from the tied row/column.
3. Allocate item to (2) as done in previous examples.
4. Repeat the processes (1) to (3) until all allocations are made.
5. Crossed out row/column cannot be used to construct penalty.

Example

Solve the transportation problem for the given data using Vogel's approximation method.

Solution:

	Lagos	**Ibadan**	**Enugu**	**Supply**	**Row penalty**
Abuja	800 400	700	600	400	100
Sokoto	1600	1000	900	1200	100
Kano	1900	1800	1200	900	600
Demand	~~1400~~ 1000	950	150		
Column penalty	800	300	300		

Table 6.1.3a

In table 6.1.3a, allocate 400 rams to cell Ln, with the least unit cost 1*800 in column 1 with the highest penalty 800. Cross out row 1 with exhausted supply from Abuja. Compute the next row/column penalties as shown in figure 6.1.3b below. Recall that Crossed out row or column cannot be used to compute penalty.

	Lagos	**Ibadan**	**Enugu**	**Supply**	**Row penalty**
Abuja	800	700	600		
	400				
Sokoto	1600	1000	900	1~~20~~0	
		950		250	100
Kano	1900	1800	1200	900	600
Demand	1000	950	150		
Column penalty	300	800	300		

Table 6.1.3b

In table 6.1.3b, allocate 950 rams to cell L_{22}, with the least unit cost ₦1000, in column 2 with highest penalty 800. Cross out satisfied column 2.

Compute the next row/column penalties as shown in figure 6.1.3c below: Recall that crossed out row/column cannot be used to compute penalty

	Lagos	**Ibadan**	**Enugu**	**Supply**	**Row penalty**
Abuja	800 400	700	600	___	___
Sokoto	1600	1000 950	900 ~~150~~	~~250~~ 100	700
Kano	1900	1800	1200	900	700
Demand	1000	-	~~150~~		
Column penal	300	-	300		

Table 6.1.3c

Observe in table 6.1.3c that row 2 and row 3 have the highest penalty of 700. Choose row 2 arbitrarily, and make the following allocation.

In table 6.1.3c allocate 150 rams to L_{23} with the least unit cost N900 in the highest row penalty. Cross out satisfied column 3 and compute the next row/column penalties as shown in table 6.1.3d below:

	Lagos	**Ibadan**	**Enugu**	**Supply**	**Row Penalty**
Abuja	800 400	700	600	___	___
Sokoto	1600 100	1000 950	900 150	900 100	100

Kano	1900 900	1800	1200	900	
Demand	1000 900	-	-		
Column penal	300	-	-		

Table 6.1.3d

In table 6.1.3d, allocate 100 rams to cell L_{21} with the least unit cost N1600 in the only column with penalty 300. Thus, row 2 is satisfied while 900 rams are left in column 1. Allocate the remaining 900 rams to cell L_{31} to exhaust the supply from Kano in row 3 and satisfy the demand in Lagos.

The associated transportation cost in table 2.6d is:

C = 400 x 800 + 100 x 1600 + 950 x 1000
+ 150 x 900 + 900 x 1900
= 320,000 + 160,000 + 950,000 + 135,000 + 1,710,000
= N3,275,000

Clearly, this is an improvement on the North – st corner and the least cost results.

6.3 Optimization Technique/optimal solution in TP

Recall that solutions obtained via the North st corner, the least cost and the Vogel's approximation methods are initial feasible solutions of the transportation problem. The next and final stage is to obtain the minimum cost of transportation. Applying the optimization technique on any of the initial feasible solutions does this.

Example

Apply the optimization technique on the initial feasible solution obtained via the least cost method to compute the minimum transportation cost for the given data.

Recall that the last table used in computing the initial feasible transportation cost via the least cost method is:

	Lagos	Ibadan	Enugu	Supply
Abuja	800	700 250	600 150	400
Sokoto	1600 500	1000 700	900	400
Kano	1900 900	1800	1200	900
Demand	1400	950	150	

Table 6.2.1a

Step 1

Compute the dispatch unit U_1 from i and the reception unit cost V_j at location j for every cell with allocation.

Thus $C_{ij} = U_i + V_j$

Where

U_i = shadow cost of dispatching a unit of the item from source i to cell L_{ij}

V_j = shadow cost receiving a unit of the item from location j to L_{ij}

C_{ij} = the cost of transporting a unit of the item from source i to location j. in cell L_{ij}.

By convention, $U_1 = 0$

For the cells with allocation in our example, obtain the following equations and hence their solutions.

For cell L_{12}.

$U_1 + V_2 = 700 \Rightarrow V_2 = 700$ since $U_1 = 0$

For cell L_{13},

$U_1 + V_3 = 600 \Rightarrow V_3 = 600$

For cell L_{22},

$U_2 + V_3 = 1000 \Rightarrow U_2 + 700 = 1000 \Rightarrow U_2 = 300$

For cell L_{21}

$U_2 + V_1 = 1600 \Rightarrow 300 + V_1 = 1600 \Rightarrow V_1 = 1300$

For cell L_{31}

$U_3 + V_1 = 1900 \Rightarrow U_3 + 1300 = 1900 \Rightarrow U_3 = 600$

Thus

$U_1 = 0, U_2 = 300 \qquad U_3 = 600$

$V_1 = 1300 \qquad V_2 = 700 \qquad V_3 = 600$

Step 2

Compute the unit shadow costs for the unallocated cells using the dispatched and reception units costs in step 1 above.

For unallocated cells:

L_{11}: $U_1 + V_1 = 0 + 1300 = 1300$

L_{23}: $U_2 + V_3 = 300 + 600 = 900$

L_{32}: $U_3 + V_2 = 600 + 700 = 1300$

L_{33}: $U_3 + V_3 = 600 + 600 = 1200$

Step 3

Compute the table of differences in unit costs by subtracting the unit shadow costs in step 2 from their corresponding actual unit costs.

If the differences are positive for all unallocated Cells, then the minimum cost has been reached. If one or more of the cells record negative difference, then the cost can be further reduced.

In our example table 6.2.1b below shows is the table of differences. In table 6.2.1b, the actual unit costs are shown in the right hand corner of the unallocated cells. The shadow costs obtained in step 2 are written in the cells while the differences are figures in circles in the cells.

	Lagos	**Ibadan**	**Enugu**
Abuja	-500 800 1300		
Sokoto		0 900	900
Kano		500 1800 1300	0 1200 1200

Table 6.2.1b

It is clear from Tale 6.2.1b that cell L_{11} has negative difference of 500. Hence the minimum cost level is yet to be reached.

Step 4

Allocation is necessary in any unallocated cell where unit shadow cost is greater than the actual unit cost. Thus, transfer of units of items from an allocated cell to cell L_{11} is required.

Where you have more than one cell in which the unit shadow cost is greater than the actual unit cost, consider the highest saving cell for transfer of units. When a tie occurs, select one of them arbitrary for transfer of units.

Recall that cell L_{11} is the only saving cell in this example. Proceed with the process of transfer of units to cell L_{11} in the initial feasible solution as reflected in table 6.2.1a as follow.

	Lagos	**Ibadan**	**Enugu**	**Supply**
Abuja	+θ	-θ 250	150	400
Sokoto	-θ 500	+θ 700		1200
Kano	900			900
Demand	1400	950	150	

Table 6.2.1c

a. Let +θ units be transferred into cell L_{11} from an allocated cell.

b. The transfer of (+θ) to cell L_{11} implies that θ must be transferred from an allocated cell in column 1 or row 1 to satisfy the supply or demand requirement. In order to ensure a complete cycle of transfer, let θ units be transferred from cell L_{21}. Hence (-θ) units is fixed in cell L_{21}.

c. Transfer (+θ) units to cell L_{22} to satisfy the supply requirement in row 2

d. Transfer (-θ) units from cell L12 to satisfy the demand requirement in column 2. Hence (-θ) units is fixed in cell L_{12}.

e. The initial (+θ) units in cell L22 and the final (-θ) unit in cell L12 cancel each other to satisfy the supply requirement in row 1.

f. From cell L12 and L21, where θ units is subtracted, it is clear that the maximum value of θ that must be subtracted to avoid a negative value in any allocated cell is minimum (250,500) = 250 units

g. Adding or subtracting θ = 250 as indicated in Table 6.2.1c give rise to the new allocation in Table 6.7d below.

	Lagos	**Ibadan**	**Enugu**
Abuja	800 250	700	600 150
Sokoto	1600 250	1000 950	900
Kano	1900 900	1800	1200

Table 6.2.1d

From table 6.7d, the new cost of transportation is

C = 250 x 800 + 150 x 600 + 250 x 1600 + 950 x 1000 + 900 x 1900

= N3,350, 000

Clearly, this is cheaper than the initial feasible cost of N347, 000 obtained via the least cost method.

Step 5

In order to find out whether or not the minimum cost of transportation is reached, repeat step 1-3 with table 6.2.1c as the initial feasible solution as follow:

Recall the last feasible solution Table 6.2.1d in Table 6.2.1e below.

	Lagos	**Ibadan**	**Enugu**	**Supply**
Abuja	800 250	700	600 150	400
Sokoto	1600 250	1000 950	900	1200
Kano	1900 900	1800	1200	900
Demand	1400	950	150	

Table 6.2.1e

Compute the dispersion and reception shadow costs from allocated cells in Table 6.2.1e as follow:

$U_1 + V_1 = 800 \Rightarrow V_1 = 800$ since $U_1 = 0$

$U_1 + V_3 = 600 \Rightarrow V_3 = 600$

$U_2 + V_1 = 1600 \Rightarrow U_2 = 1600 – 800 = 800$

$U_2 + V_2 = 1000 \Rightarrow V_2 = 1000 – 800 = 200$

$U_3 + V_1 = 1900 \Rightarrow U_3 = 1900 – 800 = 1100$

For the unallocated cells therefore, the unit shadow costs are

For cell L_{12}: $U_1 + V_2 = 200$

For cell L_{23}: $U_2 + V_3 = 800 + 600 = 1400$

For cell L_{32}: $U_3 + V_2 = 1100 + 200 = 1300$

For cell L_{33}: $U_3 + V_3 = 1100 + 600 = 1700$

Thus the differences actual unit costs and the unit shadow costs in unallocated cells are as reflected in table 6.2.1f below.

	Lagos	**Ibadan**	**Enugu**	**Supply**
Abuja	800	700 500	600	
Sokoto	1600	1000	900 -500 1400	
Kano	1800	1800 500 1300	1200 -500 1700	

Table 6.2.1f

It is clear from table 6.2.1f that cells L_{23} and L_{33} have negative difference of 500 units each. The minimum cost level is therefore yet to be reached.

Step 6

Recall the last feasible solution 6.2.1c in Table 6.2.1g below

	Lagos	**Ibadan**	**Enugu**	**Supply**
Abuja	+θ 250		-θ 150	400
Sokoto	250	950		1200
Kano	-θ 900		+θ	900
Demand	1400	950	150	

Table 6.2.1g

In Table 6.2.1g, select cell L_{33} arbitrarily for transfer of +θ units. (Recall that when there is a tie, in the maximum saving cells, select one of the affected cells arbitrary). Complete the cycle of transfer as shown in Table 6.2.1g. From table 6.2.1g, the maximum value of θ to be subtracted from cell L_{13} and L_{31} to ensure that no allocated cell record a negative value is:

θ = minimum (900,150) = 150 units. Thus, table 6.2.1h below reflects the new allocation.

	Lagos	**Ibadan**	**Enugu**
Abuja	900 400	700	600
Sokoto	1600 250	1000 950	900
Kano	1900 750	1800	1200 150

Table 6.2.1h

∴ The new transportation cost is

C = 4000 x 800 + 250 x 1600 + 950 x 100 + 750 x 1900 x 150

+ 150 x 1200

= N3,275,000

The result above is exactly the same as that obtained for the initial feasible solution via Vogels approximation approach. Hover, the figures in their cells slightly differ.

Step 7

In order to find out whether or not the minimum cost has been reached, repeat step 1-3 with Table 6.2.1i as the initial feasible solution. Recall the last feasible solution Table 6.2.1h in Table 6.2.1i below

	Lagos	**Ibadan**	**Enugu**	**Supply**
Abuja	800 400	700	600	400
Sokoto	1600 250	1000 950	900	1200
Kano	1900 750	1800	1200 150	900
Demand	1400	950	150	

Table 6.2.1i

Complete the dispersion and reception shadow costs for allocated cells in Table 6.2.1i as follows

$U_1 + V_1 = 800 \Rightarrow V_1 = 800$ since $U_1 = 0$

$U_2 + V_1 = 1600 \Rightarrow U_2 = 1600-800 = 800$

$U_3 + V_1 = 1900 \Rightarrow U_3 = 1900 - 800 = 1100$

$U_2 + V_2 = 1000 \Rightarrow V_2 = 1000 - 800 = 200$

$U_3 + V_3 = 1200 \Rightarrow V_3 = 1200 - 1100 = 1100$

$\therefore$ The unit shadow costs for the unallocated cells

L_{12}: $U_1 + V_2 = 200$

L_{13}: $U_1 + V_3 = 100$

L_{23}: $U_2 + V_3 = 800 + 100 = 900$

L_{32}: $U_3 + V_2 = 1100 + 200 = 1300$

Thus the differences actual unit costs and the unit shadow costs in unallocated cells are as reflected in table 6.2.1j below.

	Lagos	**Ibadan**	**Enugu**
Abuja		500 700 200	500 600 100
Sokoto			0 900 900
Kano		500 1800 1300	

Table 6.2.1j

It is clear from Table 6.2.1j that none of the unallocated cells has a negative difference. The minimum cost level has therefore been reached. Thus, the minimum costs is N327,5000 as obtained in step 6.

Summary for Study Session 6

In this Study Session, you have learnt that:

Transportation is centered on minimizing the cost of moving a given quantity of an item from specified sources to specified locations. A balanced transportation model is when the total quantity demanded is equal to the total quantity supplied. There are three methods to produce a starting feasible solution in a transportation problem, namely: North-st corner method, least cost method and Vogel's approximation technique. The more iteration the lesser the objective value in a transportation problem.

Self-Assessment Question 6

	1	**2**	**3**	**4**	**Supply**
A	75	60	80	55	250
B	45	70	65	70	850
C	60	50	85	75	750
D	90	70	60	85	350
E	55	65	95	80	300
Demand	150	450	900	500	

(1) Table 6.9a below indicates the number of drums of palm oil to be transported from sources A, B, C, D, and E to locations 1, 2, 3, and 4. The figures in the small boxes in the cells are the unit costs in Naira of transporting a drum of palm oil from source i to location j. Determine the total cost of transportation using the North-st Corner method. Given that 10% discount is allod.

(2)

Table 6.9a

2.

	1	**2**	**3**	**4**	**Supply**
A	25	35	20	40	250
B	45	23	38	28	400
C	36	46	44	34	100
D	48	32	42	26	550
Demand	650	50	350	250	

Table 6.10

A trader wants to transport crates of minerals from four sources A, B, C and D to four locations 1, 2, 3 and 4 as indicated in the table above. The figures in the cells are the unit costs in Naira of transporting a crate of mineral from source i to location j.

The trader approaches two transporters. The first is ready to give ten percent discount if the North st corner method is adopted in calculating the transportation cost. On the other hand, the second is willing to give five percent discount if the least cost method is adopted. Which of the two methods will benefit the trader economically?

3.

	1	**2**	**3**	**Supply**
A	15	12	18	120
B	25	16	26	400
C	24	28	22	280
Demand	350	200	250	

Table 6.11

The table above indicates some 20-kg bags of Gari to be transported from sources A, B and C to locations 1, 2 and 3. If Vogels approximation method is adopted, determine the total cost of transportation. The figures in the cells are the unit costs in Naira of transporting a 20-kig bag of Gari from source i to location j.

4. **Table 6.12**

	Saki	**Oyo**	**Ogbomoso**	**Demand**
Ibadan	35	x	40	500 + p
Lagos	70	55	65	450 + 2p
Ilorin	60	50	30	300
Supply	150	95	2p + 300	

Table 6.12 shows the supply of 30-kg bags of charcoal from Saki, Oyo and Ogbomosho and the demand for the charcoal in Ibadan, Lagos and Ilorin. The figures in the small

boxes in the cells are the unit costs in Naira of transporting a 30-kg bag of the charcoal from source i to location j. It is given that: total supply from all sources is equal to total demand in all locations. Furthermore, the total cost of transporting the charcoal from all

	Ibadan	**Lagos**	**Ilorin**	**Supply**
Kano	2x	2x+2	x+3	1500
Kaduna	x+3	2x	x+1	900
Zaria	x+2	2x-1	x	600
Demand	1200	1400	400	

sources to all locations using the North-st corner methods is N68,500. Determine:

ai. the value of P, aii. the value of x

b. the total cost of transportation using the least cost method.

5. **Table 6.13**

Table 6.13

Shows the supply of tins of ground nut oil produced by Danjumo oil merchant from Kano, Kaduna and Zaria and the demand for the oil at Ibadan Lagos and Ilorin. The expressions in the small boxes in the cells re the unit costs in Naira of transporting one tin of oil from source i to their destinations using the North st corner method is N28,400, determine:

a. (i) x (ii) the unit costs (b) the total cost of transportation using the least cost method.

6. Using the Vogel's approximation methods, find the maximum cost of the transportation problem

From/To	**Ojoo**	**Apata**	**Iwo road**	**Dugbe**	**Factory**
Kano	10	2	20	11	15
Kaduna	12	7	9	20	25
Katsina	4	14	16	18	10
Warehouse requirement	5	15	15	15	50

Where the values in the cells are the costs of transportation of the commodity.

References

Don T. Phillips, A. Ravindran, James Solberg (1976): *"Operation Research: Principles and Practice"*

Hamdy A. Taha (2007): *"Operations Research: An Introduction."* 8th edition, Upper Saddle River, New Jersey.

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Study Session Seven: Assignment Problem

Expected duration: 1 week or 2 contact hours

Introduction

This is a special Transportation problem. Recall that transportation is centered on methods of minimizing the cost of moving item from certain sources to specified locations. On the other hand, Assignment is concerned with methods of maximized the profit or minimizing the cost of executing m jobs from m sources assigned to m sources assigned to m individuals in m locations.

For example, consider the situation of assigning n workers to n jobs, consider the situation of assigning n jobs to n machines, and consider the situation of assigning n workers to n machines for a particular cost $c_{ij.}$ The objective of the model is to determine the optimum (least-cost) assignment of workers to jobs.

Learning Outcome for Study Session 7

At the end of this Study Session, you should be able to:

7.1 Differentiate between the transportation and assignment problems.
7.2 Discuss the rules of "Hungarian method" in an assignment.
7.3 Explain with illustrative examples the transportation problems.

Assumptions:

(1) The supply amount and demand amount at each cost is equal. $\Sigma a_i = \Sigma b_j = 1$
(2) No of workers equal no of jobs and if not balance can always add fictitious workers or fictitious jobs to effect this result i.e. a worker to a job.

(3)
(4) $X_{ij} = \begin{cases} 0 & \text{if the ith worker is not assigned to the jth job} \\ 1 & \text{if the ith worker is assigned to the jth job} \end{cases}$

The model is thus given as:

(5) Min $Z = \sum^{n}\sum^{n} c_{ij}x_{ij}$

s.t

$\sum_{j}^{n} x_{ij} = 1$ (i = 1, …, n)

$\sum_{i}^{n} x_{ij} = 1$ (j = I, …, n)

(6) The optimal solution of the assignment model remain the same if a constant is subtracted or added to any row (column) of the cost matrix.

(7) If one can create a new c_{ij} matrix with zero entries and if these zero elements or a subset thereof constitute a feasible solution, thus feasible solution is optimal because the cost cannot be –ve.

(8) A simple solution algorithm in solving the assignment problem is called the **HUNGARIAN METHOD**.

7.2 Hungarian Method: No of covered line < n

Step I: In each of the above tableau, locate the smallest element and subtract it from every element in that row. Repeat this procedure for each column (the column minimum is determined after the row operation). The revised cost matrix will have at least one zero, in every row and column.

Step II: Determine whether there exist a feasible assignment involving only one zero cost in the revised cost matrix function i.e. find if the revised matrix has any zero entries no two of which are in the same row/column. If such an assignment exists, go to step III.

Step III: Cover all zeros in the revised cost matrix with a few horizontal and vertical lines as possible. Each horizontal line must pass through an entire row, each vertical line must pass through an entire column.

The total no of lines in this minimal covering will be smaller than n. locate the smallest no in the cost matrix not covered by a line, subtract this number from every element not covered by the line and add it to every element covered by 2 lines. E.g. find the optimal solution to the assignment problem with the following cost matrix.

Student	P	Q	R	S	T	Test for optimality
1	1	7	0	1	5	
2	5	0	4	5	5	1 -> R 5 -> R
3	6	1	4	7	0	2 -> Q then;
4	1	4	3	1	0	3 ->T Not optimal
5	7	4	0	2	3	4 -> T

If no feasible assignment can be found among the resulting zero entries, repeat Step III. Otherwise return to Step II; for optimality.

Example:

Student	P	Q	R	S	T	Row Operation (Pi's)
1	3	9	2	3	7	2
2	6	1	5	6	6	1
3	9	1	5	6	6	3
4	2	5	4	2	1	1
5	9	6	2	4	5	2

Column Operation (qj's): (1, 0 0 1 1 0)

Student	P	Q	R	S	T	Column Operation (qj's)
1	0	7	0	0	5	1 -> P, R, S ⇒ 1 -> P
2	4	0	4	4	5	2 -> Q, ⇒ Q
3	5	1	4	6	0	3 -> T ⇒ T
4	0	4	3	0	0	4 -> P, S, T ⇒ S
5	6	4	0	1	3	5 -> R ⇒ T

It is optimal.

$Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$ = p_i's + q_j's = (2 + 1 + 3 + 1 + 2) + (1 + 0 + 0 + 1 + 0) = N11, or

Z = ₦(3 + 1 + 3 + 2 + 2) = ₦11…………***Objective value.***

Summary of Study Session 7

In this study session, you have learnt that;

1. The assignment model is a special case of the transportation model, except for the number of machines which are equal to the number of the jobs.
2. An algorithm used to solve the assignment problem is called the Hungarian method.

Self-Assessment Question 7

1. Assign the Workers to the Jobs.

Jobs

Workers	**1**	**2**	**3**	**4**	**5**
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

2. Chore/duty

Child	**M**	**P**	**W**
J	15	10	9
K	9	15	10
T	10	12	8

3. **Chore**

Child	**1**	**2**	**3**	**4**
1	1	4	6	3
2	9	7	10	9
3	4	5	11	7
4	8	7	8	5

4. Determine the assignments that will minimize the total cost. Find the minimum total cost of assignments in table 7.14

	A	**B**	**C**	**D**	**E**
Lasun	15	23	38	26	19
Bepo	26	18	21	32	36
Ropo	29	35	42	20	17
Depo	16	40	19	25	23
Kole	28	23	24	28	37

Table 7.14

5. One driver each, from six groups of companies 1, 2, 3, 4, 5 and 6 are assigned to deliver certain goods to six locations A, B, C, D, E and F. The distances in Km from the companies to the locations are as indicated in table 7.15 below. Assign jobs to the Drivers in such a way that the total distance covered by the drivers is minimum. Determine the minimum distance.

	A	B	C	D	E	F
D_1	53	80	60	45	77	53
D_2	51	72	66	50	74	53
D_3	58	75	66	48	81	56
D_4	49	76	63	45	74	57
D_5	56	71	65	49	79	52
D_6	58	76	64	49	77	55

Table 7.15

6. A car rental firm assigns 4 of its drivers in 4 of its locations A.B.C. and D to 4 service centers P.Q.R. and S. The service charges in N'00 are as indicated in table 7.16 below. Assign jobs to drivers to maximize returns. Determine the total maximum return.

	P	Q	R	S
A	25	19	24	17
B	15	23	15	22
C	15	20	18	26
D	20	25	16	22

7. The profits in million naira accrued to a road construction company by assigning five road construction works to four field engineers A, B, C, D, for supervision are as indicated in the cells of the table below.

Field Engr.	**1**	**2**	**3**	**4**	**5**
A	5	7	6	3	8
B	12	8	7	9	15

C	10	4	9	6	12
D	6	14	12	16	9

Table 7.17

Assign engineers to jobs in a way to maximize profit.

8. Four coordinators from 4 government establishments are assigned to 5 coordinating centers to coordinate 5 government-sponsored workshops for one ek. The figures in table 7.18 below are the coordinators expenses in N'000 for the duration of the workshop. Assign jobs to coordinators in a way to minimize the total cost.

	A	**B**	**C**	**D**	**E**
Chukwu Emeka	250	315	290	270	285
Abubakar	245	295	260	325	300
Adewale	305	265	320	290	285
Ambrose	275	280	310	260	295

Table 7.18

References

Don T. Phillips, A. Ravindran, James Solberg (1976): *"Operation Research: Principles and Practice"*

Hamdy A. Taha (2007): *"Operations Research: An Introduction."* 8th edition, Upper Saddle River, New Jersey.

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Study Session Eight: Network Analysis I

Expected duration: 1 week or 2 contact hours

Introduction

A network is a graph which consists of a number of nodes or junction points each joined to some or all of the others by arcs or links or lines. A network is a graph such that a flow can take place in the branches of the graph. A network may or may not be oriented (orientation information, profit etc.) examples of network include road networks, liquid networks.

Learning Outcomes for Study Session 8

At the end of this Study Session, you should be able to:

8.1 Define some concepts in a network analysis.
8.2 Highlight rules in drawing a network (graph).
8.3 Discuss extensively maximal flows in network analysis.
8.4 Explain with illustrative examples, the shortest routes in network analysis

8.1 Definitions of Some Concepts in Network Analysis

Capacity Restriction: This is when there is a limit to the magnitude of flows in any branch or node of a network. If there is a limit to the magnitude of flow in any branch or node of the network, it implies that the capacity of the network is restricted.

Source Node: Every branch having this node as an end point is oriented in such a way that flow in the branch moves away from the node.

Sink Node: Every branch having this node as an end point is oriented in such a way that flow in is directed from other nodes to this node. In a network it is assumed that the branches are connected only at the nodes.

8.2 Rules for Drawing Network

The following are the rules for drawing a network:

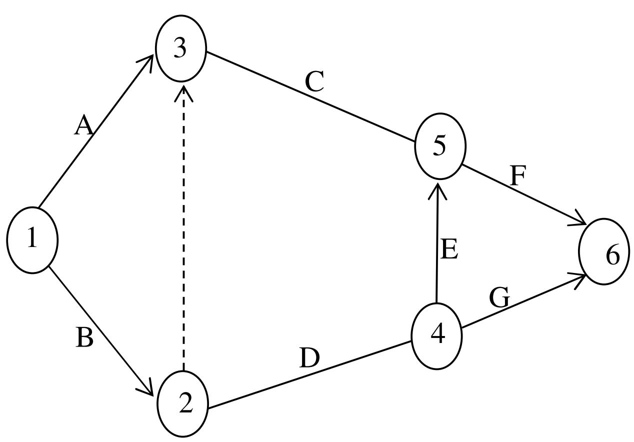
1. Each activity in the project is represented by one and only one arrow or arc.
2. No two activities can be identified by the same head or tail event. For example, In building electrification and fixing of doors can be performed concurrently, each of these activities must be represented by different branch.
3. In order to ensure the correct precedence relationship in a network the following questions must be answered as every activity and events is added to the network.
   i. What activity or event can be completed immediately before this activity can start What activity must follow this immediately
   ii. What activity must occur concurrently with this activity?

Example

Construct a network diagram for a project consisting of the following activities

Activity	Immediate Predecessor(s)
A	-
B	-
C	A, B
D	B
E	D
F	C, F
G	D

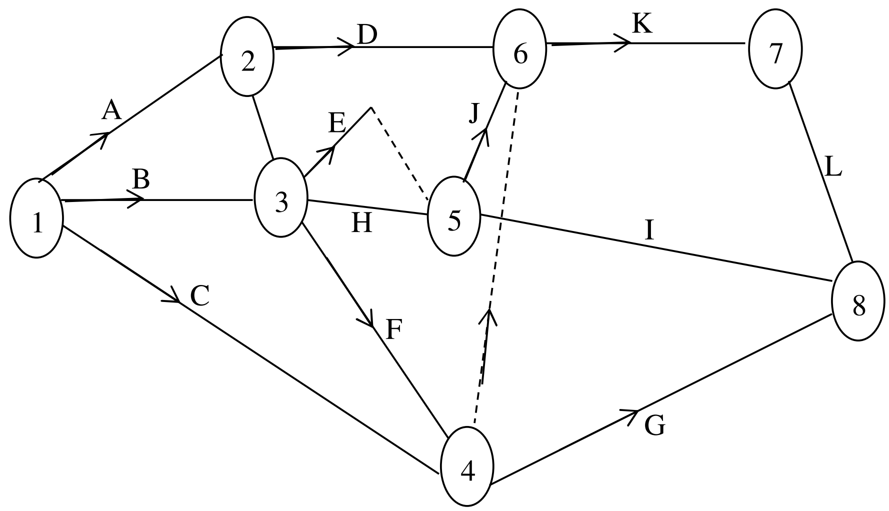
F and G are the terminal activities of the project



Example

Construct a network diagram that comprises activities A to L with this precedence relationship.

1. AB & C are the initial activities. They can start simultaneously
2. AB must proceed D
3. B precedes E, F, & H
4. F & C precede G
5. E & H precedes I & J
6. C, D, F & J precedes K
7. K precedes L
8. I, G & L are the terminal activities



8.3 Maxima Flow in Network

Consider a network with a single source, a single sink and some intermediate nodes. The objective in maxima flow is to develop a shipping plan or schedule that maximizes the amount of materials sent between 2 points. The point of Origin is called the SOURCE and the destination is called the SINK. Various shipping lanes or routes that exist which link the source and sink directly or via intermediate locations called JUNCTIONS. It is assumed that junctions cannot store materials i.e. any materials arriving at a unction is shipped immediately to another location.

The remaining assumptions are:

1. The network is oriented.
2. There is capacity restriction on each branch (for example $d_{ij} \geq 0$ on branch (ij)
3. No capacity restrictions on the nodes. In any branch i,j the flow can assume any value X_{ij} satisfying $0 \leq X_{ij} \leq d_{ij}$

4. There is conservation of flow at any node, other than source and sink i.e. Total inflow = Total outflow

A flow X_{ij} from node i to j is said to be feasible if there is conservation of flow at every node except the source and sink, and if

$$0 \leq X_{ij} \leq d_{ij} \text{ for every } i,j$$

A feasible flow X_{ij} in a network is said to be maxima if the amount flowing from source to sink is **finite** and if there is no other feasible flow which yields a larger flow from source to sink

Determination of Maxima Flows

Given a network with a capacity of each arc of branch labeled perform the following algorithm:

Start at the source and move along branches with the capacity until a path is determined from source to sink.

Search the path identified for the branch i the smallest capacity say k and set the flow in this path equal to k

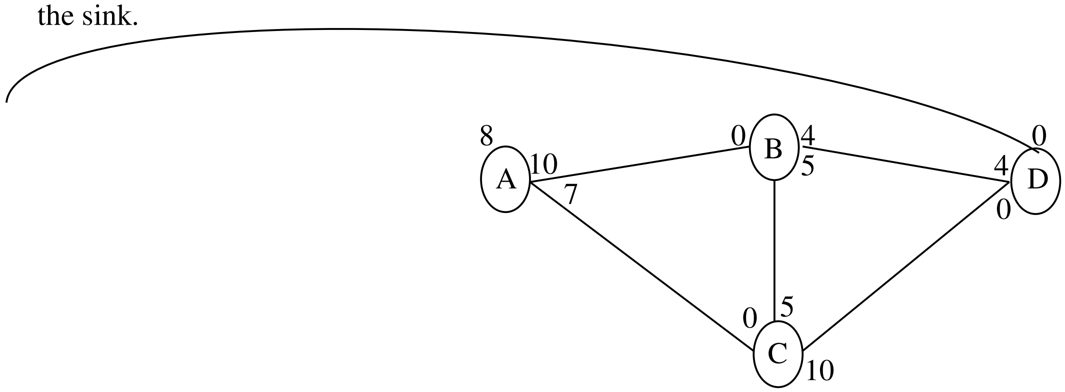
Then decrease by k the capacity of each branch in the path and increase the reverse capacity by k.

Go to Step 1.

The maxima flow is then the sum of the flows in the paths which re obtained at each step. The optimal shipping schedule is determined by comparing the original network with the final network. Any reduction in capacity signifies a shipment.

Example

Solve the maxima flow problem for the network shown below if A is the source and D is the sink.



8.4 Shortest Route Algorithm

In a network with positive flow may want to determine the shortest route between a source and a destination. To do this, use the shortest Route Algorithm as follows:

Step 1: Construct a master list by tabulating under each nodes in ascending order of cost the branch incident on that node. Each branch under a given node is written with that node as its first node. Omit from the list nodes having the source as its 2nd node or having th sink as its first node.

Step 2: Star (*) the source and assign it value zero (0), locate the cheapest branch incident on the source and circle it. Star (*) the 2nd node of this branch and assign this node a value equal the cost of distance of the branch. Delete from the master list all other branches that have the newly started (*) node as 2nd node.

Step 3: If the newly starred node is the sink go to step 5. Otherwise so to step 4

Step 4: Consider all starred nodes having uncircled branches under them in the current master list. Of each one, add the value assigned to the node the cost of distance of the cheapest uncircled branch under it.

Denote the smallest of these sum as n and circle the branch whose cost contributed to n. Star the 2nd node of this branch and assign it the value n. delete from the master list all other branches having the newly starred node as 2nd node, then go to step 3.

Step 5: Assign the value z* to the sink. A minimum cost or route is obtained recursive beginning with the sink by including in the path each circled branch whose 2nd node belongs to the path.

Example

An individual who lives in city R in a state works in another city W, wants a car route that will minimize the early morning driving time. This man has recorded driving times (mins) along major highways between different intermediate cities, these data are shown in the table below:

	R	**C**	**O**	**T**	**P**	**W**
R	-	18	-	32	-	-
C	18	-	12	28	-	-
O	-	12	-	17	-	32
T	32	28	17	-	4	17
P	-	-	-	4	-	11
W	-	-	32	17	11	-

Solution:

R	C	O	T	P	W
RC = 18	CO = 12	OC = 12	TP = 4	PT = 4	~~WP = 11~~
RT = 32	CB = 18	OT = 17	TO = 17	PW = 11	~~WT = 17~~
	CT = 28	OW = 32	TW = 17		WO = 32
			TC = 28		
			~~TR = 32~~		

Omit where R is 2nd and where W is 1st and follow the procedures above, then have the final table.

R*(0)	C*(18)	O* (30)	T* (32)	P* (36)	W* (47)
RC = 18	CO =12	~~OC = 12~~	~~TP = 4~~	~~PT = 4~~	
CT = 32	~~CT = 28~~	~~OT = 17~~	~~TW = 17~~	PW = 11	
			~~OW = 32~~	~~TO = 17~~	
				~~TC = 28~~	

R → T → P W = 47 (**This is the shortest route for the above problem**)

Summary of Study Session 8

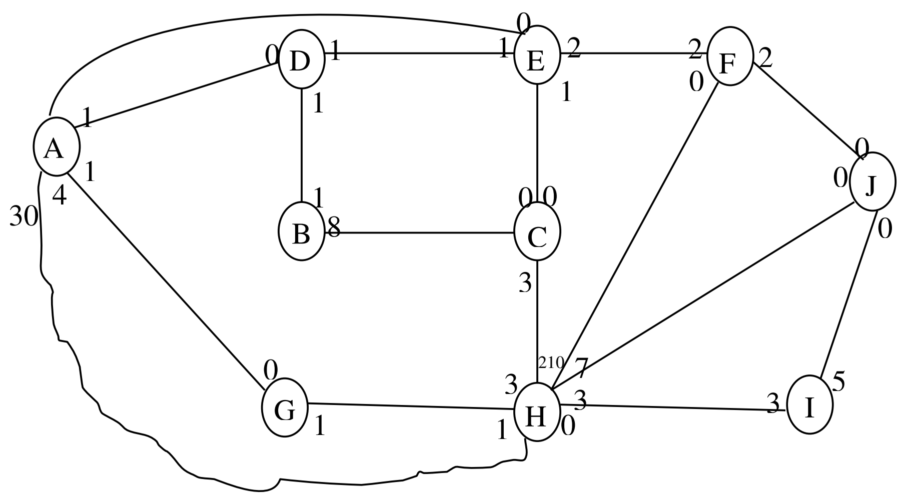
In this study session, you have learnt that:

1. Network analysis (NA) is a graph which consists of a number of nodes or junction points each joined to some or all of the others by arcs or links or lines.
2. Maxima flows and shortest routes are very important aspect of an network analysis. The objective in maxima flow is to develop a shipping plan or schedule that maximizes the amount of materials sent between two points, while in the other hand.
3. The shortest route in a network is used to determine with the positive flow the closest or shortest path between a source and a destination.

Summary of Study Session 8

	Park Entrance	Wild falls	Majestic Rock	Sunset point	The Meadow
Park Entrance	…	7.1	19.5	19.1	25.7
Wild Falls	7.1	…	8.3	16.2	13.2
Majestic Rock	19.5	8.3	…	18.1	5.2
Sunset point	19.1	16.2	18.1	…	17.2
The Meadow	25.7	13.2	5.2	17.2	…

2.



Find the max-flow and the shipping schedule in the above network

3. The ABC Manufacturing Company is considering the construction of a new factory building. The following list shows the project activities, precedence relationships, and time estimates:

Activity	**Description**	**Immediate Predecessor(s)**
A	Problem definition	-
B	Preliminary study of costs and constraints	A
C	Analysis of problems in existing building	A
D	Incorporation of requirements in new building	C
E	Detailed drawings of new building	B, C
F	Contractor building a prototype	D, E
G	Cost analysis	E
H	Engineers reviewing feasibility	G
I	Contractor building the factory	G, F
J	Building inspection	I, H
K	Final plant layout	J

Develop a CPM network for this project.

4.

Activity	**Predecessor**
A	-
B	-
C	A,B
D	B
E	D
F	C,E
G	D

Draw the project network for the above problem.

References

Don T. Phillips, A. Ravindran, James Solberg (1976): *"Operation Research: Principles and Practice"*

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Study Session Nine: Inventory Theory and Applications

Expected duration: 1 week or 2 contact hours

Introduction

Successful, ll-organized business rely heavily on inventory management systems (modules) to make certain that adequate inventory levels are on hand to satisfy their customer demand. Inventory is concerned with usable resources at the same time idle, such resources are; machines, men, money or materials. Inventory is called 'stock' if the resources involved are material. It is the stock of any item kept to maximize the output of an organization.

Inventory control is therefore the act of monitoring movement of items from the point of receipt into the warehouse (store) to the point of issue to the customers (users) so as to minimize, if not to prevent theft, stock –outs and deterioration thus maximization of profits. The main reason of taking inventory control into consideration is that it determines profitability

Learning Outcomes for Study Session

At the end of this module, you should be able to;

9.1 Define some concepts or terminologies of inventory.

9.2 Explain inventory and list types of inventory costs.

9.3 Discuss Pareto or ABC Analysis.

9.4 Explain Re-order level system of control.

9.1 Simple Useful Terminologies of Inventory Control

1. **Total Cost:** This is the combination of ordering cost and holding cost, it consists of all the amount of what it takes to produce any commodity. Note that the overall objectives of inventory control are the maintaining stock levels so that total costs are at minimum. This is done by considering (i) when to order or demand (ii) how many to order or demand.
2. **Demand:** this is the total number of products required in a period. It is usually exposed per annum.

3. **Physical stock:** This is the quantity of items physically in stock at a particular time.
4. **Re-order quantity:** This is quantity required as the replenishment order.
5. **Periodic order System:** It places minimal emphasis on record keeping. Hover a risk of substantial over stock or under stock may arise unless inventories are checked for assurance that the model is still appropriate.
6. **Free balance:** It is physical stock-pulls, expected replenishment orders minus unfulfilled orders.
7. **Lead or procurement Time:** This is the period of time between placing an order and when the goods are received.
8. **Re-order level:** Is the level of stock at which a further replenishment order should be placed. It can be obtained by maximum consumption times maximum lead time.
9. **Economic order quantity (EOQ):** This is the size of the quantity to be ordered (demanded) to minimize total cost.
10. **Safety stock (SS):** This is a "Cashion" of inventory held to mitigate the uncertainties of forecast and lead times. Higher safety stock levels increase the likelihood that goods are available, but also drive up inventory level and costs.
11. **Buffer stock:** This is a stock allowance that kept to cover emergencies, or unexpected increase in demand during lead time. It is sometimes refer to as minimum stock or safety stock.

Reasons For Maintaining Inventory

It is very important to maintain an inventory in any establishment or enterprises because of the following reasons.

1. It reduces production cost
2. it prepares for possible shortage in future particularly during inflation period
3. It provides services to the customer at a short notice
4. It ensures a smooth and efficient flow of production process.
5. It helps in absorbing seasonal variations in usage or demand
6. It may earn price discount because of large purchase
7. Inventory avoid stock out.

9.2 Inventory costs

This is the cost of holding goods in stock. It is expressed usually as a percentage of the inventory value; it includes capital, warehousing, depreciation, insurance, taxation, obsolescence and shrinkage costs.

Types of inventory costs

There are three main types of costs in inventory. They are as follows:

1. Ordering costs, manufactory costs or set-up cost: this is a fixed cost incurred each time in order to place to replace inventory irrespective of the size of order components of ordering cost.

These include:

- Transport costs
- Costs associated with clearing of imported goods
- Administrative and clerical costs;
- Production costs e.g. invoice précising cost.
- Cost of receiving or inspecting the items.

2. Holding cost (storage cost) or sometimes called carrying cost. This is cost of keeping items in inventory. These include:

- Warehouse equipment maintenance and running cost
- Audit and stock –taking costs
- Handling costs
- Interest on capital invested on the stock
- Warehousing expenses, salaries and wages of warehouse staff.

3. Stock cost (shortage penalty or shortfall cost). It is also known as stock-out cost: This is the costs of keeping good will of customer or avoiding loss of future sales. This is the main aim of inventory control. Theses include:

- Cost of emergency replenishment
- Lost of contribution due to loss of prospective customer
- Loss of future sales due to loss of sale.
- Shortage of raw materials leads to stoppage of production and this frustrate labour force.
- Customers may take alternative product as substitutes.

9.3 Pareto Analysis

This is a statistical technique in decision making that is used for selection of a limited number of tasks that produce significant overall effect. It uses the Pareto principle the idea that by doing 20% of work, you can generate 80% of the advantage of doing the entire job. Or in terms of quality improvement, a large majority of problems (80%) are produced by a few causes (20%). Pareto analysis can be used to identify cost drivers that are responsible for the majority of cost incurred by ranking the cost drivers in order of value.

Pareto analysis is a formal technique useful where many possible causes of action are competing for your attention. In essence, the problem -solver estimates the benefit delivered by such action, then selects a reasonably close to the maximal possible one.

Pareto analysis is a creative way of looking at cause of problems because it helps stimulate thinking and organize thoughts. Hover, it can be limited by its exclusion of possibly important problems which may be small initially, but which grow with time. It should be combined with other analytical tools such as failure mode and effects analysis and fault tree analysis for examples.

Steps to identify the important causes using Pareto analysis

1. form a table listing the causes and their frequency as a percentage
2. Arrange the rows in the decreasing order of importance of the causes (i.e the most important causes first).
3. Add a cumulative percentage column to the table
4. Plot with causes on x-axis and commulative percentage on y-axis.
5. join the above points to form a curve
6. Plot (on the same graph) a bar graph with causes on x-axis and percent frequency on y-axis.
7. Draw line at 80% on y-axis parallel to x-axis. Then drop the line at the point of intersection with the curve on x-axis.

   This point on the x-axis separate the important causes (on the left) and trivial causes (on the right).
8. Review the chart to ensure you are capturing at least 80% of the causes.

Another illustration

Class A Items 80% of value in 20% of items

Class B items -15% of value in 30% of items –Regular review

Class C items – 5% value in 50% of items in frequent review.

Such review helps to ensure that resources are used to maximum advantage.

9.4 Re-order level system of control

This is a system that maintains the level of stock at which a further replenishment order should be placed. It is a fixed quantity system (at variable times) which minimizes the likelihood of a stock out. It can be obtained by maximum consumption multiplied by maximum lead time. But, Re-order quantity is the quantity required as the replenishment order. It is also known as the "two-bin system". This is one of the basic inventing control systems.

Characteristics of Re-order level System

i. A predetermined re-order level is set for each item.

ii. When the stock level falls to the re-order, a replenishment order is used.

iii. The replenishment order quantity is invariably the EOQ

iv. The name 'two-bin system' comes from the simplest method of operating the system whereby the stock is segregated into two bins. Stock is initially drawn from the first bin and a replenishment order issued when it becomes empty.

v. Most organizations operating the re-order level system maintain stock records with calculated re-order levels which trigger off the required replenishment order.

Example 9.5.1a

Given the following data relate to a particular stock item.

Normal usage	110 per day
Minimum usage	50 per day
Maximum usage	140 per day
Lead time	25-30 days
EOQ (previously calculated)	5,000

Using this data to calculate the various control levels

Solution:

Re-order level = maximum usage x maximum lead time

= 140 x 30

= 4,200 units

Minimum level = re-order level – Average usage for Average lead time

= 4,200 – (110 x 27.5)

= 1175 units

Maximum level = Re-order level + EOQ – minimum Anticipated usage lead time.

= 4,200 + 5,000 – (50 x 25)

= 7,950 units

Note: The re-order level is calculated so that if the worst anticipated position occurs, stock would be replenishment in time.

Maximum level is calculated so that management will be warned when demand is the minimum anticipated and consequently the stock level is likely to rise above the maximum intended.

The minimum level is calculated so that management will be warned when demand is above average and accordingly buffer stock is being used. There may be no danger, but the situation needs watching.

Advantages of Re-order Level System

i. Lor stocks on average
ii. Items ordered in economic quantities via the EOQ calculation.
iii. Somewhat more responsive to fluctuations in demand
iv. Automatic generation of a replenishment order at the appropriate time by comparison of stock level against re-order level.
v. Appropriate for widely differing types of inventory within the same firm.

Disadvantages of Re-order Level System

i. Many items may reach re-order level at the same time, thus overloading the re-ordering system.
ii. Items come up for re-ordering in a random so that there is no set sequence.

iii. In certain circumstances (e.g. variable demand, ordering costs, etc.), the EOQ calculation may not be accurate.

Periodic Review system (Basic inventory control system): This system is sometimes called constant cycle system. It is a classic independent inventory system which its inventory level is received at regular time intervals (e.g. 3 days, two eks etc.).

Characteristics of Periodic Review System

i. Stock levels for all parts are revised at fixed intervals e.g. every fortnight.
ii. Where necessary a replenishment order is issued
iii. The quantity of the replenishment order is not a previously calculated EOQ, but is based upon; the likely demand until the next review, the present stock level and the lead time.
iv. The replenishment order quantity seeks to bring up to a predetermined level.
v. The effect of the system is to order variable quantities at fixed intervals as compared with the re-order system, where fixed quantities are ordered at variable intervals

Advantages of Periodic Review System

i. All stock items are reviewed periodically so that there is more chance of obsolete items being eliminated.
ii. Economics is placing orders may be gained by spreading the purchasing office load more evenly.
iii. Larger quantity discounts may be obtained when a range of stock items are ordered at the same time from a supplier.
iv. Because orders will always be in the same sequence, there may be production economies due to more efficient production planning being possible and lor set up costs. This is often a major advantage and results in the frequent use of some form of periodic review system in production control system in firms where there is a preferred sequence of manufacture, so that full advantage can be gained from the predetermined sequence implied by the periodic review system.

Disadvantages of Periodic review System

i. In general larger stocks are required as re-order quantities must take account of the period between reviews as ll as lead times.
ii. Re-order quantities are not at the optimum level of a correctly calculated EOQ.

iii. Less responsive to changes in consumption. If the rate of usage change shortly after a review, a stock out may ll occur before the next review.
iv. Unless demands are reasonably consisted, it is difficult to set appropriate periodic for review.

Economic Order Quantity (EOQ)

This is the number of units that a company should add to inventory with each order to minimize the total costs of inventory; such as holding costs, order costs and shortage costs. The EOQ is used as part of a continuous review inventory system, in which the level of inventory is monitored at all times, and a fixed quantity is ordered each time the inventory level reaches a specific re-order point. The EOQ provides a model for calculating the appropriate re-order point and the optimal re-order quantity to ensure the instantaneous replenishment of inventory with no shortages. It can be a valuable tool for small business owners who need to make decisions about how much inventory to keep on hand, how many items to order reach time, and how often to re-order to incur the lost possible costs.

The EOQ mode assumes that demand is constant and that inventory is depleted at a fixed rate until it reaches zero. At that point, a specific number of items arrive to return the inventory to its beginning level. Since the model assumes instantaneous replenishment, there are no inventory shortages or associated costs. Therefore, the cost of inventory under the EOQ model involves a tradeoff between inventory holding costs and order costs.

Ordering a large amount at one time will increase a small business's holding costs, while making more frequent orders of fer items will reduce holding costs but increase order costs. The EOQ will sometimes change as a result of quantity discounts, which are provided by some suppliers as an incentive for customers to place larger orders.

9.4.1 Formula approach to EOQ

It is an inventory-related equation that determines the optimum order quantity that a company should hold in its inventory given a set cost of production, demand rate and other variables. This is done to minimize variable inventory costs. Due to the variations in price relative to time, the following assumptions are made in calculating EOQ.

1. The ordering cost must be known and constant.
2. The stock-holding costs must be known and constant.
3. The annual demand for item is known with certainty.

4. The purchase price per unit is constant i.e. no quantity discount.
5. The replenishment is made instantaneously i.e., the whole batch is delivered at once (lead time is zero, i.e. fixed).
6. Stock-out are not allod.

Derivation of EOQ Formula

Let D = demand per annum

C = ordering cost per order

Q = quantity ordered

h = holding cost (carrying) per annum

r = production rate per unit time

s = stock out costs per item per annum

a. D/Q = Number of times order is made per annum

b. $Qh/2$ = annual average holding cost

c. $Q/2$ = Average stock

d. CD/Q = annual ordering cost.

∴

i. Total cost (TC) = annual ordering cost + annual average holding cost

$$TC = \frac{CD}{Q} + \frac{Qh}{2}$$

The quantity which makes the total cost to be minimized is obtained by differentiating "TC' with respect to Q and equating the derivation to zero.

$$\frac{dTC}{dQ} = -\left(\frac{CD}{Q2}\right) + \frac{h}{2} = 0$$

$$\frac{h}{2} = \frac{CD}{Q2} \Rightarrow Q^2 = \frac{2CD}{h}$$

$$\therefore \quad Q = \sqrt{\frac{2CD}{h}} = EOQ$$

ii. EOQ with gradual replenishment (EOQr) is

$$EOQ_r = \sqrt{\frac{2CD}{h\left(1-\frac{D}{r}\right)}}$$

iii. EOQ where stock-out are permitted and stock out costs are known

$$\text{EOQs} = \sqrt{\frac{2CD}{h}} x \sqrt{\frac{h+s}{S}}$$

Example

A company uses 50,000 widgets per animal which are ₦10 each .The ordering and handling cash are ₦ 150 per order and carrying cash are 15% of purchase price per annum, i.e it cash ₦1.50 per annum to any a widget in struck (₦10 x 15%). Calculate the Economic order quantity of the company.

Solution:

Base: EOQ formula

$$\text{EOQ} = \frac{\sqrt{2cD}}{h}$$

Where C = Ordinary cost = ₦150

D= Demand rate per annum = 50,000

A= Carrying costs (holding costs) per annum = ₦10x10%

= 1.5 per annum (widget)

$$\therefore \quad EOQ \quad = \quad \sqrt{\frac{2\,x\,150\,x\,50000}{1.5}} \quad = \sqrt{10{,}000{,}000}$$

$\approx$ 3,162 widgets

Example

If all other data are assumed to be the same in example 5.7.1.1a, but the company has decided to make the widgets in its own factory. The necessary machinery has been purchased which has a capacity of 250,000 widgets per annum.

$$\text{EOQ (with gradual replenishment)} = \sqrt{\frac{2CD}{h\left(1-\frac{D}{r}\right)}}$$

Where r= 250,000, D= 50,000, C= 150, h= 1.5

$$\therefore \quad EOQ_2 \quad = \quad \sqrt{\frac{2\,x\,150\,x\,50000}{1.5\left(1-\frac{50000}{250000}\right)}} \quad = \quad \sqrt{\frac{300\,x\,50000}{1.5\left(0.8\right)}}$$

$EOQ_r \approx$ 3,535 Widgets

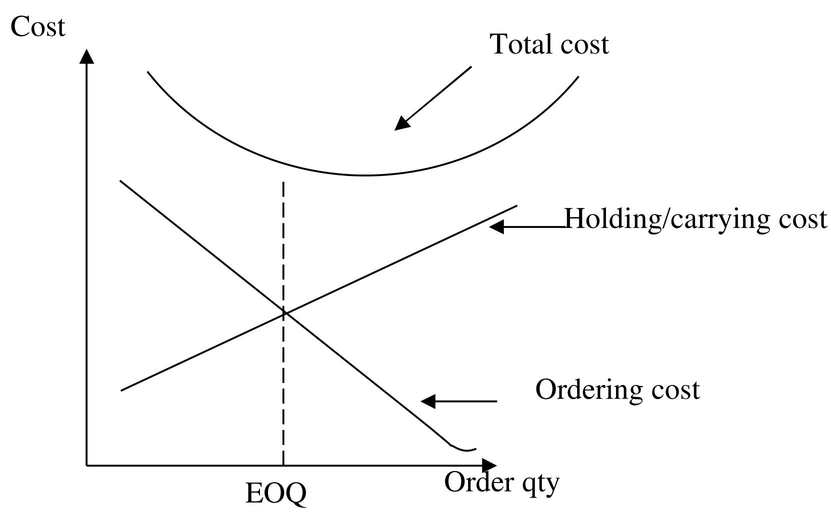
Note: This is larger because the usage during the replenishment period has the effect of loring the average stock holding costs.

Graphical approach to EOQ

This is the graphical representation or relationship between the three costs; total cost, holding (carrying) cost and ordering cost. The point (size) of the quantity at which the total costs is at its minimum, which is also the point at which the holding and ordering costs intersect on the graph is called the Economic order quantity (EOQ).

The cost is plotted on the y-axis against the order size (quantity) on the x-axis

The sketch is as follows:



Example Using example above, find the EOQ

Using, graphical method; Hint: assumes the quantity order of (100-600)

Solution:

The following calculations are necessary for the graphical solution.

Total costs per annum= ordering cost per annum + holding cost per annum.

Ordering cost per annum = Average number of orders per annum $\left(\frac{D}{Q}\right)$ × ordering

Cost per order (c) Average number of orders per annum = $\frac{\text{Annual demand}}{\text{Order quantity}}$

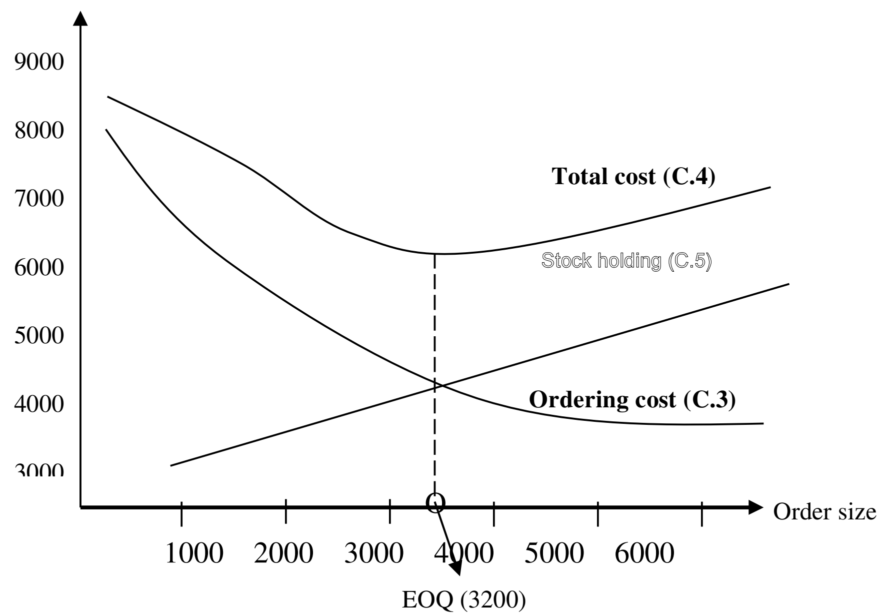
Carrying (holding) cast per annum = average stock level ($Q/2$) X carrying cost per order (h) (purchase price). $= Q\,{}^{h}/_{2}$

Based on the above principles, the following table gives the costs various order quantities.

Table Ordering and stock holding costs for various order quantities.

C. 1 **Order quantity (Stock quantity)**	**C . 2** **Average number of order per annum (₦)** $(50000 \div c.1)$	**C. 3 Annual ordering** $(c.2 x N 150)$	**C.4** **Average stock(₦)** **(C.1/2-)**	**C. 5** **Stock holding (₦) cost per annum (C.4 x ₦1.5)**	**C.6 total cost (₦)** **C.3 + C.5)**
1000	50	7500	500	750	8250
2000	25	3750	1000	1500	5250
3000	$16\,^2/_3$	2500	1500	2250	4750
4000	$12^1/_2$	1875	2000	3000	4875
5000	10	1500	2500	3750	5250
6000	$8^1/_3$	1250	3000	4500	5750

From the table (5.7.2), can sketch the graph of these costs as follows:



From the graph the following are observed

i. The EOQ is approximately 3200 widgets which means that an average of slightly under 16 orders will have to be placed a year.

ii. The bottom of the total cost curve is relatively flat, indicating that the exact value of the EOQ is not critical. This is typical of most EOQ problems.

iii. A closed accurate value cannot be obtained but approximately equal to the exact value {3162 widgets} gotten from the formula method.

Summary of Study Session 9

In this Study Session, have learnt the following:

1. Investment control is basically adopted by companies in order to reduce their total cost and thus increase their profits.
2. There are three main types of costs in inventory, namely: ordering costs, carrying or holding costs and stock or shortfall costs.
3. Parents analysis is a formal technique useful where many possible causes of action are competing for your attention.
4. There are basically two main inventory control systems namely re-order level and periodic review systems.

Self Assessment Question to Study Session 9

1a. How many main types of cost do have in inventory?

b. Explain the types mentioned in (a) above

c. Distinguish between re-order level and periodic review systems of control in inventory.

2a. Explain Pareto analysis, with examples

2b. The following data relate to a given stock item

Normal usage	1300 per day
Minimum usage	900 per day
Maximum usage	2000 per day
Lead time	15-20 days
EOQ	30,000

Calculate the various control Levels; Re-order level, minimum level and maximum level

3a. What is Economic order quantity (EOQ)?

b. A firm uses 20,000 cartons of chemical per annum. It costs N9.00 to place an order for the chemical. The unit cost of the chemical is N50.00 and it takes 8% of the cost to keep the chemical in stock. Calculate:

i. The economic order quantity (EOQ)

ii. The annual ordering cost

iii. The total annual cost

iv. Use graphical approach to determine the EOQ. You can assume the quantity to be 100-700 cartons.

v. The EOQ if shock out is allow, and the cost per chemical per annum is N2.50.

4

a. A company uses 100,000 units per year which cost N3 each carrying cost are 1% per month and ordering costs are N250 per order. What is the EOQ?
b. What would be the EOQ if the company made the items themselves on a machine with a potential capacity of 600,000 units per year?

5. Demand is 5,000 units per year. Ordering costs are N100 per order and the basic unit price is N5. Carrying costs are 20% per annum. Discounts are available thus:

1,200	-	1,399	less 10%
1,400	-	1,499	less 15%
1,500 and over			less 20%

What is the most economical quantity to order?

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Study Session Ten: Principles of Sequencing and Scheduling

Expected duration: 1 week or 2 contact hours

Introduction

Principles of Sequencing (Project Planning) and Scheduling strikes a unique balance between theory and practice, providing an accessible introduction to the concepts, methods, and results scheduling theory and its core topics. Project is an expensive venture which takes a great deal of efforts and time. Because of its nature, it has to be properly planned to optimally utilize the limited resources.

You can imagine yourself building a house, constructing a bridge or even planning your academic career. How long will each of these projects take you? Can you guess?' "With necessary information at your disposal about the activities involved and their time elements, you can, through the knowledge of project analysis, which shall explore in this Study Session.

Learning Outcomes for Study Session 10

At the end of this module, you should be able to know:

10.1 Outline the various steps involved in planning and controlling a project.
10.2 Analyze the methods of project planning

10.1 Illustrations of project planning and scheduling

First, you need to recall the salient characteristics; of a project: A project is a task of considerable size that requires a great deal of investment in time and effort. It is a one-short of activities, which have definite beginning and ending point. It is a unique undertaking usually with long-term completion period e.g. construction of third mainland bridge in Lagos.

Though some of the operations or activities involved may be run concurrently, their degree of irreversibility is none or minor. Projects can be classified based on sectors (public and private), time horizon or size.

From the nature of the project, you can see that it requires proper planning and scheduling for effective control. Project planning involves all managerial activities necessary in structuring a course of action.

On the other hand, project scheduling is concerned with determining the many activities required by the project, the precedence relationship of these activities, time requirement and the best sequence for executing the project. Project planning and scheduling aims at optimum utilization of resources and maintenance of delivery promises.

10.1.1 Procedures in Project Planning and Control

You need to consider the following general steps in project planning and scheduling,

i. Create a project-planning group.
ii. State project objectives.
iii. Enumerate all activities that must be done to achieve project objectives.
iv. Sequence the activities enumerated.
v. Allocate such resources as material equipment and manpower to each of the proposed project activities.
vi. Estimate the time and cost required.
vii. Revise the plan until- acceptable total plan is evolved.
viii. Develop an organization that can implement, monitor and control the project plan.

10.2 Methods of project planning and control

There are two methods of project planning and control open to you. These are traditional (Gantt Chart) and modern (Network Analysis) techniques,

The Gantt Chart

The Gantt Chart method is the oldest method for coping with the problem of project planning, Developed by Henry L. Gantt, it is known as Gantt Project Planning Chart, A Gantt chart is a bar chart that systematically shows the relationship of activities over some time period.

A typical Gantt chart lists project activities along the Y-axis and time phase requirement along the X-axis. For instance, let us consider some activities involved in a building project and how they will appear on a Gantt chart.

Building Project Month

	Project activity	**1**	**2**	**3**	**4**	**5**	**6**
1	Buy the land						
2	Obtain survey plane						
3	Obtain Certificate of Occupancy						
4	Draw up building plan						
5	Obtain approval of building plan						
6	Engage building contractor						
7	Start the building						

Advantages

i. The method at least provides a means of organizing our thinking.

ii. It enables us to visualize the precedence relationship of each activity and serves as control means (i.e. to know if are working to schedule as specified in the chart).

Disadvantages

i. It requires a great amount of trial and error time

ii. The solution provided is sub-optimal especially for a complex project.

Network Analysis

The Network Analysis is a modern method which breaks a complicated project into a set of individual activities and places them in a logical network (see the introduction in Study Session 8 and the detail shall be discussed later).

Self-Assessment for Study Session 10

1. What do you understand by Sequencing and Scheduling in Operations Research?
2. Outline the procedures needed in Project Planning and Control
3. Highlight the two methods use in Project Planning and Control
4. In question three above state two advantages and disadvantages for each of them, mentioned.

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